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Abstract

An adequate and efficient modelling of non-linear functions is one of the principal difficulties in applying linear programming to real-life optimization problems. Here we present a few approaches for such modelling, particularly representing disjunction, conjunction and sign-based Boolean functions.

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ABSTRACT
An adequate and efficient modelling of non-linear functions is one of the principal difficulties in applying linear programming to real-life optimization problems. Here we present a few approaches for such modelling, particularly representing disjunction, conjunction and sign-based Boolean functions.

KEYWORDS
Linear optimization; Modelling techniques; Boolean functions.

Theorem 1. $\forall i \quad 1 \le i \le I$ such that $i, I \in \mathbb{N}$; $I \ge 2$ and $x_i, X \in \{0,1\}$:
An inequality

$$\frac{1}{I}\sum_{i=1}^{I} x_i \le X \le \sum_{i=1}^{I} x_i \tag{1}$$

is equivalent to the equality $X = \vee_{i=1}^{I} x_i$

Proof. Follows from Lemma 1 and Lemma 2. □

Lemma 1. $\forall i \quad 1 \le i \le I$ such that $i, I \in \mathbb{N}$; $I \ge 2$ and $x_i, X \in \{0,1\}$:
If inequality (1)

$$\frac{1}{I}\sum_{i=1}^{I} x_i \le X \le \sum_{i=1}^{I} x_i$$

is valid, then

$$X = \vee_{i=1}^{I} x_i$$

Proof. Let us consider two complementary cases:

$$\text{Case 1: } \forall i \quad 1 \le i \le I \quad i \in \mathbb{N} \quad x_i = 0 \tag{2}$$

$$\text{Case 2: } \exists j \quad 1 \le j \le I \quad j \in \mathbb{N} \quad x_j = 1 \tag{3}$$

In Case 1, from (2) it follows that $\sum_{i=1}^{I} x_i = 0$, which in turn means that $\frac{1}{I}\sum_{i=1}^{I} x_i = 0$. Then according to (1), $0 \le X \le 0$ which means that $X = 0$. But from the assumption of the case, it also holds that $\vee_{i=1}^{I} x_i = 0$ – therefore $X = \vee_{i=1}^{I} x_i$.

In Case 2, from (3) it follows that $1 \le \sum_{i=1}^{I} x_i \le I$ and therefore, $0 < \frac{1}{I} \le \frac{1}{I}\sum_{i=1}^{I} x_i \le 1$. Combining this with Equation (1) and the fact that $X \in \{0,1\}$, we obtain that $X = 1$. Additionally, *as* $\vee_{i=1}^{I} x_i = 1$, therefore, also in this case, $X = \vee_{i=1}^{I} x_i$.

Therefore, in all cases, $X = \vee_{i=1}^{I} x_i$. □

Lemma 2. $\forall i \quad 1 \le i \le I$ such that $i, I \in \mathbb{N}$; $I \ge 2$ and $x_i, X \in \{0,1\}$
If

$$X = \vee_{i=1}^{I} x_i$$

then inequality (1)

$$\frac{1}{I}\sum_{i=1}^{I} x_i \le X \le \sum_{i=1}^{I} x_i$$

is valid.

Proof. Again, we explore two complementary cases:

$$\text{Case 1: } \forall i \quad 1 \leq i \leq I \quad i \in \mathbb{N} \quad x_i = 0 \tag{4}$$

$$\text{Case 2: } \exists j \quad 1 \leq j \leq I \quad j \in \mathbb{N} \quad x_j = 1 \tag{5}$$

In Case 1, from (4) follows that $\sum_{i=1}^{I} x_i = 0$ and consequently $\frac{1}{I}\sum_{i=1}^{I} x_i = 0$. According to definition of X and (4), $X = 0$. Therefore inequality (1) is valid in Case 1.

In Case 2, from (5) follows that $\sum_{i=1}^{I} x_i \geq 1$ and $\frac{1}{I}\sum_{i=1}^{I} x_i > 0$. According to definition of X and (5), $X = 1$. Therefore inequality (1) is valid for Case 2 as well.

Hence, in all cases, inequality (1) holds. □

Theorem 2. $\forall x, X, s \in \mathbb{Z}$ such that $X >= 0; \quad 0 \leq x \leq X; \quad s \in \{0, 1\}:$
An equality

$$s = sign(x) \tag{6}$$

is equivalent to a double inequality

$$s \leq x \leq s \cdot X \tag{7}$$

Proof. Follows from Lemma 3 and Lemma 4. □

Lemma 3. $\forall x, X, s \in \mathbb{Z}$ such that $X >= 0; \quad 0 \leq x \leq X; \quad s \in \{0, 1\}:$
If the equality in Equation (6)

$$s = sign(x)$$

holds, then the inequality in Equation (7)

$$s \leq x \leq s \cdot X$$

is valid.

Proof. Let us consider two complementary cases that comply with the definition of x:

$$\text{Case 1:} \quad x = 0 \tag{8}$$

$$\text{Case 2:} \quad x > 0 \tag{9}$$

In Case 1, from Equation (6) it follows that $s = sign(0) = 0$, thus, the inequality in Equation (7) holds

$$0 \leq x \leq 0 \cdot X$$

.

Since $x \in \mathbb{Z}$, Case 2 inequality in Equation (9) can be equivalently rewritten as $1 \leq x$. On the other hand, $s = sign(x) = 1$, hence, the inequality in Equation (7) holds

$$1 \leq x \leq 1 \cdot X$$

Therefore, in all the cases the inequality in Equation (7) is valid. □

Lemma 4. $\forall x, X, s \in \mathbb{Z}$ such that $X >= 0; \quad 0 \leq x \leq X; \quad s \in \{0, 1\}:$
If the inequality in Equation (7)

$$s \leq x \leq s \cdot X$$

is valid, then the equality in Equation (6)

$$s = sign(x)$$

holds.

Proof. Again, we explore two complementary cases constructed based on the definition of x, that were presented in Equation (8) and Equation (9):

$$\text{Case 1:} \quad x = 0$$

$$\text{Case 2:} \quad x > 0$$

In Case 1, the inequality in Equation (7) can be reduced to

$$s \leq 0 \leq s \cdot X$$

In this case, the only acceptable value of $s \in \{0, 1\}$ would be

$$s = 0 = sign(0) = sign(x)$$

Thus, the equality in Equation (6) holds in Case 1.

Since $x \in \mathbb{Z}$, Case 2 inequality in Equation (9) can be equivalently rewritten as

$$1 \leq x$$

Since $s \in \{0, 1\}$, the only value

$$s = 1 = sign(x)$$

would make the inequality in Equation (7) valid

$$1 \leq x \leq X$$

Hence, the equality in Equation (6) holds in Case 2 as well. □

Theorem 3. $\forall i \quad 1 \leq i \leq I$ such that $i, I \in \mathbb{N}$; $I \geq 2$ and $x_i, X \in \{0,1\}$

The inequality

$$-\frac{I-1}{I} + \frac{1}{I}\sum_{i=1}^{I} x_i \leq X \leq \frac{1}{I}\sum_{i=1}^{I} x_i \tag{10}$$

is equivalent to the equality

$$X = \wedge_{i=1}^{I} x_i$$

Proof. Follows from Lemma 5 and Lemma 6. □

Lemma 5. $\forall i \quad 1 \leq i \leq I$ such that $i, I \in \mathbb{N}$; $I \geq 2$ and $x_i, X \in \{0,1\}$

If inequality (10)

$$-\frac{I-1}{I} + \frac{1}{I}\sum_{i=1}^{I} x_i \leq X \leq \frac{1}{I}\sum_{i=1}^{I} x_i$$

is valid, then

$$X = \wedge_{i=1}^{I} x_i$$

Proof. Let us consider two complementary cases:

$$\text{Case 1: } \forall i \quad 1 \leq i \leq I \quad i \in \mathbb{N} \quad x_i = 1 \tag{11}$$

$$\text{Case 2: } \exists j \quad 1 \leq j \leq I \quad j \in \mathbb{N} \quad x_j = 0 \tag{12}$$

In Case 1, from (11) it follows that $\sum_{i=1}^{I} x_i = I$ and consequently $\frac{1}{I}\sum_{i=1}^{I} x_i = 1$, $-\frac{I-1}{I} + \frac{1}{I}\sum_{i=1}^{I} x_i = \frac{1}{I} > 0$. Via substitution to (10) we then obtain $0 < X \leq 1$, which means that $X = 1$. Additionally, it holds that $\wedge_{i=1}^{I} x_i = 1$ – therefore $X = \wedge_{i=1}^{I} x_i$.

In Case 2, from (12) it follows that $\sum_{i=1}^{I} x_i < I$ and consequently $0 \leq \frac{1}{I}\sum_{i=1}^{I} x_i < 1$, $-\frac{I-1}{I} + \frac{1}{I}\sum_{i=1}^{I} x_i \leq 0$. Via substitution to (10) we obtain $0 \leq X < 1$, which means that $X = 0$. Additionally it holds that $\wedge_{i=1}^{I} x_i = 0$ – therefore $X = \wedge_{i=1}^{I} x_i$.

Therefore, in all cases, $X = \wedge_{i=1}^{I} x_i$. □

Lemma 6. $\forall i \quad 1 \leq i \leq I$ such that $i, I \in \mathbb{N}$; $I \geq 2$ and $x_i, X \in \{0,1\}$

If

$$X = \wedge_{i=1}^{I} x_i$$

then inequality (10)

$$-\frac{I-1}{I} + \frac{1}{I}\sum_{i=1}^{I} x_i \leq X \leq \frac{1}{I}\sum_{i=1}^{I} x_i$$

is valid.

Proof. Let us consider two complementary cases:

$$\text{Case 1: } \forall i \quad 1 \le i \le I \quad i \in \mathbb{N} \quad x_i = 1 \tag{13}$$

$$\text{Case 2: } \exists j \quad 1 \le j \le I \quad j \in \mathbb{N} \quad x_j = 0 \tag{14}$$

In Case 1, from (13) it follows that $X = 1$, $-\frac{I-1}{I} + \frac{1}{I}\sum_{i=1}^{I} x_i = \frac{1}{I} < 1$, and $\frac{1}{I}\sum_{i=1}^{I} x_i = 1$. Therefore (10) in this case is valid.

In Case 2, from (14) it follows that $X = 0$, $-\frac{I-1}{I} + \frac{1}{I}\sum_{i=1}^{I} x_i \le 0$, and $0 \le \frac{1}{I}\sum_{i=1}^{I} x_i \le 1$. Therefore (10) is valid for this case as well.

Therefore inequality (10) holds in all cases. □

Theorem 4. $\forall i \quad 1 \leq i \leq I$ such that $i, I \in \mathbb{N}$; $I \geq 2$ and $x_i, y, Z \in \{0,1\}$:
The inequality

$$\frac{1}{2} \times (-\frac{I-1}{I} + \frac{1}{I} \times \sum_{i=1}^{I} x_i + y) - \frac{1}{2 \times I} < Z \leq \frac{1}{I} \times \sum_{i=1}^{I} x_i + y \tag{15}$$

is equivalent to the equality

$$Z = (\wedge_{i=1}^{I} x_i) \vee y \tag{16}$$

Proof. Follows from Lemma 7 and Lemma 8. □

Lemma 7. $\forall i \quad 1 \leq i \leq I$ such that $i, I \in \mathbb{N}$; $I \geq 2$ and $x_i, y, Z \in \{0,1\}$:
If inequality (15)

$$\frac{1}{2} \times (-\frac{I-1}{I} + \frac{1}{I} \times \sum_{i=1}^{I} x_i + y) - \frac{1}{2 \times I} < Z \leq \frac{1}{I} \times \sum_{i=1}^{I} x_i + y$$

is valid, then equality (16)

$$Z = (\wedge_{i=1}^{I} x_i) \vee y$$

holds.

Proof. For the sake of brevity, we denote the left-hand expression and the right-hand expression of the double inequality (15) as L and R respectively:

$$L = \frac{1}{2} \times (-\frac{I-1}{I} + \frac{1}{I} \times \sum_{i=1}^{I} x_i + y) - \frac{1}{2 \times I} \tag{17}$$

$$R = \frac{1}{I} \times \sum_{i=1}^{I} x_i + y \tag{18}$$

Let us consider two complementary cases:

$$\text{Case 1: } \forall i \quad 1 \leq i \leq I \quad i \in \mathbb{N} \quad x_i = 1 \tag{19}$$

$$\text{Case 2: } \exists j \quad 1 \leq j \leq I \quad j \in \mathbb{N} \quad x_j = 0 \tag{20}$$

In Case 1, from (19) it follows that $\sum_{i=1}^{I} x_i = I$ and consequently

$$\frac{1}{I} \sum_{i=1}^{I} x_i = 1 \tag{21}$$

Hence, from Equation (17)

$$
\begin{aligned}
L = \frac{1}{2} \times (-\frac{I-1}{I} + 1 + y) - \frac{1}{2 \times I} = \\
\frac{1}{2} \times (\frac{-I+1+I}{I} + y) - \frac{1}{2 \times I} = \\
\frac{1}{2} \times (\frac{1}{I} + y) - \frac{1}{2 \times I}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\forall i \quad 1 \leq i \leq I \quad i \in \mathbb{N} \quad x_i = 1 \\
&L = \frac{1}{2} \times (\frac{1}{I} + y) - \frac{1}{2 \times I}
\end{aligned} \tag{22}
$$

From Equation (18) and Equation (21) we get

$$
R = 1 + y \tag{23}
$$

We can substitute the left-hand side and the right-hand side of the double inequality (15) with the right-hand sides of Equation (22) and Equation (23) respectively:

$$
\begin{aligned}
&\forall i \quad 1 \leq i \leq I \quad i \in \mathbb{N} \quad x_i = 1 : \\
&\frac{1}{2} \times (\frac{1}{I} + y) - \frac{1}{2 \times I} < Z \leq 1 + y
\end{aligned} \tag{24}
$$

Inside Case 1, we can consider two complementary subcases:

$$
\begin{aligned}
&\forall i \quad 1 \leq i \leq I \quad i \in \mathbb{N} \quad x_i = 1 \\
&\text{Case 1.0:} \quad y = 0 \\
&\text{Case 1.1:} \quad y = 1
\end{aligned} \tag{25}
$$

In Case 1.0, we can rewrite Equation (24) by substituting y with 0:

$$
\begin{aligned}
\frac{1}{2} \times (\frac{1}{I} + 0) - \frac{1}{2 \times I} < Z \leq 1 + 0 \qquad \iff \\
0 < Z \leq 1
\end{aligned} \tag{26}
$$

By the definition, Z is a binary value $Z \in \{0, 1\}$, therefore, Equation (26) specifies that $Z = 1$. Notice, that

$$
\begin{aligned}
&\forall i \quad 1 \leq i \leq I \quad i \in \mathbb{N} \quad x_i = 1, \quad y = 0 \\
&(\wedge_{i=1}^{I} x_i) \vee y = 1 \vee 0 = 1
\end{aligned} \tag{27}
$$

Therefore, in Case 1.0, $Z = (\wedge_{i=1}^{I} x_i) \vee y = 1$ and Lemma 7 is valid.

In Case 1.1, we substitute y with 1 in Equation (24):

$$\begin{aligned}\frac{1}{2} \times (\frac{1}{I} + 1) - \frac{1}{2 \times I} &< Z \leq 1 + 1 && \Longleftrightarrow \\ \frac{I+1}{2 \times I} - \frac{1}{2 \times I} &< Z \leq 2 && \Longleftrightarrow \\ \frac{1}{2} &< Z \leq 2 && (28)\end{aligned}$$

Since Z can have only two possible values, 0 or 1, Equation (28) specifies that $Z = 1$. Given that

$$\begin{aligned}&\forall i \quad 1 \leq i \leq I \quad i \in \mathbb{N} \quad x_i = 1, \quad y = 1 \\ &(\wedge_{i=1}^{I} x_i) \vee y = 1 \vee 1 = 1 \qquad (29)\end{aligned}$$

$Z = (\wedge_{i=1}^{I} x_i) \vee y = 1$. Hence, in Case 1.1, Lemma 7 holds as well.

In Case 2, from Equation (20) we know that $0 \leq \sum_{i=1}^{I} x_i < I$ and consequently

$$0 \leq \frac{1}{I} \sum_{i=1}^{I} x_i < 1 \qquad (30)$$

From Equation (17) and Equation (30) we get the following bounds on the left-hand expression of the double inequality (15) marked as L.

$$\begin{aligned}\frac{1}{2} \times (-\frac{I-1}{I} + 0 + y) - \frac{1}{2 \times I} &\leq L < \frac{1}{2} \times (-\frac{I-1}{I} + 1 + y) - \frac{1}{2 \times I} && \Longleftrightarrow \\ \frac{1}{2} \times (-\frac{I-1}{I} + y) - \frac{1}{2 \times I} &\leq L < \frac{1}{2} \times (\frac{-I+1+I}{I} + y) - \frac{1}{2 \times I} && \Longleftrightarrow \\ \frac{1}{2} \times (-\frac{I-1}{I} + y) - \frac{1}{2 \times I} &\leq L < \frac{1}{2} \times (\frac{1}{I} + y) - \frac{1}{2 \times I} && (31)\end{aligned}$$

To construct the bounds for the right-hand expression of the double inequality (15) (marked as R) we use Equation (18) and Equation (30):

$$\begin{aligned}0 + y \leq &R < 1 + y \\ y \leq &R < 1 + y \qquad (32)\end{aligned}$$

Inside Case 2, we consider the following complementary subcases:

$$\begin{aligned}&\exists j \quad 1 \leq j \leq I \quad j \in \mathbb{N} \quad x_j = 0 \\ &\text{Case 2.0:} \quad y = 0 \\ &\text{Case 2.1:} \quad y = 1 \qquad (33)\end{aligned}$$

In Case 2.0, we substitute y with its value 0 in Equation (31) to get the bounds on the left-hand side L of the double inequality (15):

$$\begin{aligned}\frac{1}{2}\times(-\frac{I-1}{I}+0)-\frac{1}{2\times I}\leq L<\frac{1}{2}\times(\frac{1}{I}+0)-\frac{1}{2\times I} &\quad\Longleftrightarrow\\ \frac{-I+1-1}{2\times I}\leq L<\frac{1}{2\times I}-\frac{1}{2\times I} &\quad\Longleftrightarrow\\ -\frac{1}{2}\leq L<0 &\qquad (34)\end{aligned}$$

For the right-hand side R of the double inequality (15), we substitute y with 0 in Equation (32):

$$\begin{aligned}0\leq &R<1+0\\ 0\leq &R<1 \qquad (35)\end{aligned}$$

From Equation (34) $L\in[-\frac{1}{2},0)$ and from Equation (35) $R\in[0,1)$ (see Figure 1). From the double inequality (15) $Z\in(L,R]$, hence its value should be somewhere on the right side of 0 (included) and on the left side of 1 (excluded). Given that by definition $Z\in\{0,1\}$, the only value that meets for all the constraints is $Z=0$. Notice, that the value $Z=1$ violates Equation (35) ($R\in[0,1)$) and the double inequality (15) ($Z\in(L,R]$). By checking Equation (16):

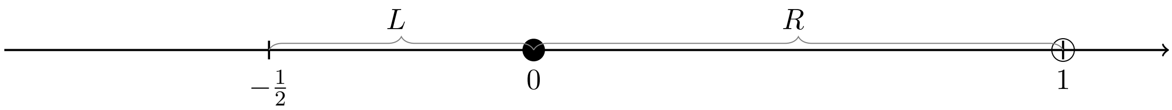

Figure 1. Determining the value of Z in Case 2.0

$$\begin{aligned}&\exists j\quad 1\leq j\leq I\quad j\in\mathbb{N}\quad x_j=0,\quad y=0\\ &(\wedge_{i=1}^{I}x_i)\vee y=0\vee 0=0 \qquad (36)\end{aligned}$$

$Z=\wedge_{i=1}^{I}x_i)\vee y=0$, hence Lemma 7 holds in Case 2.0.

In Case 2.1, y is substituted with 1. By doing this in Equation (31) we get the bounds for the left-hand side L of the double inequality (15):

$$\begin{aligned}\frac{1}{2}\times(-\frac{I-1}{I}+1)-\frac{1}{2\times I}\leq L<\frac{1}{2}\times(\frac{1}{I}+1)-\frac{1}{2\times I} &\quad\Longleftrightarrow\\ \frac{-I+1+I}{2\times I}-\frac{1}{2\times I}\leq L<\frac{I+1}{2\times I}-\frac{1}{2\times I} &\quad\Longleftrightarrow\\ 0\leq L<\frac{1}{2} &\qquad (37)\end{aligned}$$

For the right-hand side R of the double inequality (15) we substitute y with 1 in Equation (32):

$$\begin{aligned}y\leq &R<1+y\\ 1\leq &R<2 \qquad (38)\end{aligned}$$

From Equation (37) $L \in [0, \frac{1}{2})$ and from Equation (38) $R \in [1, 2)$ (see Figure 2). According to the double inequality (15) $Z \in (L, R]$, the value of Z has to be somewhere on the right side of $\frac{1}{2}$ (excluded) and on the left side of 1 (included). By definition, Z can be either 0 or 1, therefore, $Z = 1$ is the only value that can satisfy all the constraints. Notice, that the value $Z = 0$ is not eligible since it violates Equation (37) ($L \in [0, \frac{1}{2})$) and the double inequality (15) ($Z \in (L, R]$). Let us check Equation (16):

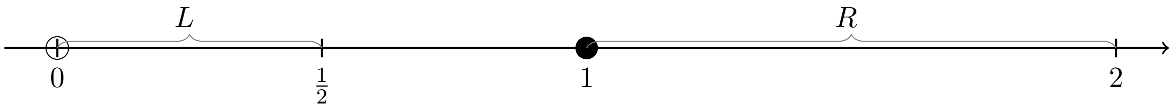

Figure 2. Determining the value of Z in Case 2.1

$$\begin{aligned} &\exists j \quad 1 \leq j \leq I \quad j \in \mathbb{N} \quad x_j = 0, \quad y = 1 \\ &(\wedge_{i=1}^{I} x_i) \vee y = 0 \vee 1 = 1 \end{aligned} \tag{39}$$

$Z \in \{0, 1\}$, from double inequality (15) $Z \in (L, R]$ Hence, Lemma 7 is also valid in Case 2.1.

Thus, we showed that Lemma 7 holds in all subcases within Case 1 and Case 2. □

Lemma 8. $\forall i \quad 1 \leq i \leq I$ such that $i, I \in \mathbb{N}$; $I \geq 2$ and $x_i, y, Z \in \{0, 1\}$:
If the equality (16)

$$Z = (\wedge_{i=1}^{I} x_i) \vee y$$

holds, then the inequality (15)

$$\frac{1}{2} \times (-\frac{I-1}{I} + \frac{1}{I} \times \sum_{i=1}^{I} x_i + y) - \frac{1}{2 \times I} < Z \leq \frac{1}{I} \times \sum_{i=1}^{I} x_i + y$$

is valid.

Proof. Let us consider the boolean expression $(\wedge_{i=1}^{I} x_i) \vee y$ in the right-hand side of Equation (16) as a disjunction of the boolean expression $\wedge_{i=1}^{I} x_i$ and the binary variable y, for the sake of applying Theorem 1. In the formulation of the theorem, we substitute X with Z and $I = 2$ terms of the disjunction $x_1,\ x_2$ with $\wedge_{i=1}^{I} x_i$ and y, hence the following bounds on the boolean expression $(\wedge_{i=1}^{I} x_i) \vee y$ are derived:

$$\frac{1}{2} \times \Big((\wedge_{i=1}^{I} x_i) + y\Big) \leq Z \leq (\wedge_{i=1}^{I} x_i) + y \tag{40}$$

Notice, that according to Theorem 3

$$-\frac{I-1}{I} + \frac{1}{I} \sum_{i=1}^{I} x_i \leq \wedge_{i=1}^{I} x_i \leq \frac{1}{I} \sum_{i=1}^{I} x_i \tag{41}$$

subject to the substitution of X with $\wedge_{i=1}^{I} x_i$ in the formulation of the theorem.

Let us consider the left-hand inequality of the double inequality (40)

$$\frac{1}{2} \times \Big((\wedge_{i=1}^{I} x_i) + y \Big) \leq Z \tag{42}$$

and the left-hand inequality of the double inequality (41)

$$-\frac{I-1}{I} + \frac{1}{I}\sum_{i=1}^{I} x_i \leq \wedge_{i=1}^{I} x_i \tag{43}$$

The right-hand side of Equation (43) appears in the left-hand side of Equation (42). Therefore, we can substitute the right-hand side of Equation (43) into the left-hand side of Equation (42):

$$\frac{1}{2} \times (-\frac{I-1}{I} + \frac{1}{I}\sum_{i=1}^{I} x_i + y) \leq \frac{1}{2} \times \Big((\wedge_{i=1}^{I} x_i) + y \Big) \leq Z \qquad \Longleftrightarrow$$

$$\frac{1}{2} \times (-\frac{I-1}{I} + \frac{1}{I}\sum_{i=1}^{I} x_i + y) \leq Z \tag{44}$$

By subtracting a positive number ($\frac{1}{2\times I}$) from the left-hand side of Equation (44) we can make the corresponding non-strict inequality strict:

$$\frac{1}{2} \times (-\frac{I-1}{I} + \frac{1}{I}\sum_{i=1}^{I} x_i + y) - \frac{1}{2 \times I} < Z \tag{45}$$

Let us consider the right-hand inequality of the double inequality (40)

$$Z \leq (\wedge_{i=1}^{I} x_i) + y \tag{46}$$

and the right-hand inequality of the double inequality (41)

$$\wedge_{i=1}^{I} x_i \leq \frac{1}{I}\sum_{i=1}^{I} x_i \tag{47}$$

The right-hand side of Equation (47) can be found in the left-hand side of Equation (46). Thus, we substitute the right-hand side of Equation (47) into the left-hand side of Equation (46).

$$Z \leq (\wedge_{i=1}^{I} x_i) + y \leq \frac{1}{I}\sum_{i=1}^{I} x_i + y \qquad \Longleftrightarrow$$

$$Z \leq \frac{1}{I}\sum_{i=1}^{I} x_i + y \tag{48}$$

One can combine the inequality (45) and the inequality (48) into a double inequality

$$\frac{1}{2} \times (-\frac{I-1}{I} + \frac{1}{I}\sum_{i=1}^{I} x_i + y) - \frac{1}{2 \times I} < Z \leq \frac{1}{I}\sum_{i=1}^{I} x_i + y \tag{49}$$

which is exactly the same as Equation (15) in the formulation of Lemma 8. □