A Response-Time Analysis for Non-Preemptive Job Sets under Global Scheduling

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Abstract

An effective way to increase the timing predictability of multicore platforms is to use non-preemptive scheduling. It reduces preemption and job migration overheads, avoids intra-corecache interference, and improves the accuracy of worst-case execution time (WCET) estimates. However, existing schedulability tests for global non-preemptive multiprocessor scheduling are pessimistic, especially when applied to periodic workloads. This paper reduces this pessimism by introducing a new type of sufficient schedulability analysis that is based on an exploration of the space of possible schedules using concise abstractions and state-pruning techniques. Specifically, we analyze the schedulability of non-preemptive job sets (with bounded release jitter and execution time variation) scheduled by a global job-level fixed-priority (JLFP) scheduling algorithm up on an identical multicore platform. The analysis yields a lower bound on the best-case response-time (BCRT) and an upper bound on the worst-case response time (WCRT) of the jobs. In an empirical evaluation with randomly generated workloads, we show that the method scales to 30 tasks, a hundred thousand jobs (per hyperperiod), and up to 9 cores.
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Abstract
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1 Introduction

While modern multicore platforms offer ample processing power and a compelling price/performance ratio, they also come with no small amount of architectural complexity. Unfortunately, this complexity—such as shared caches, memory controllers, and other shared micro-architectural resources—has proven to be a major source of execution-time unpredictability, and ultimately a fundamental obstacle to deployment in safety-critical systems.

In response, the research community has developed a number of innovative approaches for managing such challenging hardware platforms. One particularly promising approach explored in recent work [1, 15, 24] is to split each job into three distinct phases: (i) a
dedicated memory-load or prefetching phase, which transfers all of a job’s required memory from the shared main memory to a core-local private cache or scratchpad memory; followed by (ii) the actual execution phase, in which the job executes non-preemptively and in an isolated manner without interference from the memory hierarchy as all memory references are served from a fast, exclusive private memory, which greatly enhances execution-time predictability; and finally (iii) a write-back phase in which any modified data is flushed to main memory. As a result of the high degree of isolation restored by this approach [20], a more accurate worst-case execution time (WCET) analysis becomes possible since the complete mitigation of inter-core interference during the execution phase allows existing uniprocessor techniques [25] to be leveraged. Recent implementations of the idea, such as Tabish et al.’s scratchpad-centric OS [24], have shown the phased-execution approach to indeed hold great promise in practice.

From a scheduling point of view, however, the phased-execution approach poses a number of difficult challenges. As jobs must execute non-preemptively—otherwise prefetching becomes impractical and there would be only little benefit to predictability—the phased-execution approach fundamentally requires a non-preemptive real-time multiprocessor scheduling problem to be solved. In particular, Alhammad and Pellizzoni [1] and Maia et al. [15] considered the phase-execution model in the context of non-preemptive global scheduling, where pending jobs are allocated simply to the next available core in order of their priorities.

Crucially, to make schedulability guarantees, Alhammad and Pellizzoni [1] and Maia et al. [15] rely on existing state-of-the-art analyses of global non-preemptive scheduling as a foundation for their work. Unfortunately, as we show in Sec. 6, this analytical foundation—i.e., the leading schedulability tests for global non-preemptive scheduling [4, 10, 11, 13]—suffers from substantial pessimism, especially when applied to periodic hard real-time workloads.

This paper. To attack this analysis bottleneck, we introduce a new, much more accurate method for the schedulability analysis of finite sets of non-preemptive jobs under global job-level fixed-priority (JLFP) scheduling policies. Our method, which can be applied to periodic real-time tasks (and other recurrent task models with a repeating hyperperiod), is based on a novel state-space exploration approach that can scale to realistic system parameters and workload sizes. In particular, this work introduces a new abstraction for representing the space of possible non-preemptive multiprocessor schedules and explains how to explore this space in a practical amount of time with the help of novel state-pruning techniques.

Related work. Global non-preemptive multiprocessor scheduling has received much less attention to date than its preemptive counterpart. The first sufficient schedulability test for global non-preemptive scheduling was proposed by Baruah [4]. It considered sequential sporadic tasks scheduled with a non-preemptive earliest-deadline-first (G-NP-EDF) scheduling algorithm. Later, Guan et al. [10, 11] proposed three new tests; one generic schedulability test for any work-conserving global non-preemptive scheduling algorithm, and two response-time bounds for G-NP-EDF and global non-preemptive fixed-priority (G-NP-FP) scheduling. Recently, Lee et al. [13, 14] proposed a method to remove unnecessary carry-in workload from the total interference that a task suffers. These tests for sporadic tasks have been used in various contexts such as the schedulability analysis of periodic parallel tasks with non-preemptive sections [21] and systems with shared cache memories [26] or with transactional memories [1, 24]. However, these tests become needlessly pessimistic when applied to periodic tasks as they fail to discount many execution scenarios that are impossible in a periodic setting. Moreover, these tests do not account for any release jitter that may arise due to timer inaccuracy, interrupt latency, or networking delays.

To the best of our knowledge, no exact schedulability analysis for global job-level fixed-
priority non-preemptive scheduling algorithms (including G-NP-EDF and G-NP-FP) either for sporadic or for periodic tasks has been proposed to date. The exact schedulability analysis of global preemptive scheduling for sporadic tasks has been considered in several works [3, 5, 6, 9, 23]. These analyses are mainly based on exploring all system states that can be possibly reached using model checking, timed automata, or linear-hybrid automata. These works are inherently designed for a preemptive execution model, where no lower-priority task can block a higher-priority one, and hence are not applicable to non-preemptive scheduling. The second limitation of the existing analyses is their limited scalability. They are affected by the number of tasks, processors, and the granularity of timing parameters such as periods. For example, the analysis of Sun et al. [23] can only handle up to 7 tasks and 4 cores, while the solution by Guan et al. [9] is applicable only if task periods lie between 8 and 20.

In our recent work [16], we have introduced an exact schedulability test based on a schedule-abstraction model for uni-processor systems executing non-preemptive job sets with bounded release jitter and execution time variation. By introducing an effective state-merging technique, we were able to scale the test to task sets with more than 30 tasks or about 100000 jobs in their hyperperiod for any job-level fixed-priority scheduling algorithm. The underlying model and the test’s exploration rules, however, are designed for, and hence limited to, uniprocessor systems and cannot account for any scenarios that may arise when multiple cores execute jobs in parallel.

Contributions. In this paper, we introduce a sufficient schedulability analysis for global job-level fixed-priority scheduling algorithms considering a set of non-preemptive jobs with bounded release jitter and execution time variation. Our analysis derives a lower bound on the best-case response time (BCRT) and an upper bound on the worst-case response time (WCRT) of each job, taking into account all uncertainties in release and execution times. The proposed analysis is not limited to the analysis of periodic tasks (with or without release jitter), but can also analyze any system with a known job release pattern, e.g., bursty releases, multi-frame tasks, or any other application-specific workload that can be represented as a recurring set of jobs.

The analysis proceeds by exploring a graph, called schedule-abstraction graph, that contains all possible schedules that a given set of jobs may experience. To render such an exploration feasible, we aggregate all schedules that result in the same order of start times of the jobs and hence significantly reduce the search space of the analysis and makes it independent from the time granularity of the timing parameters of the systems. Moreover, we provide an efficient path-merging technique to collapse redundant states and avoid non-required state explorations. The paper presents an algorithm to explore the search space, derives merge rules, and establishes the soundness of the solution.

2 System Model and Definitions

We consider the problem of scheduling a finite set of non-preemptive jobs \( J \) on a multicore platform with \( m \) identical cores. Each job \( J_i = ([r_{i_{\min}}, r_{i_{\max}}], [C_{i_{\min}}, C_{i_{\max}}], d_i, p_i) \) has an earliest-release time \( r_{i_{\min}} \) (a.k.a. arrival time), latest-release time \( r_{i_{\max}} \), absolute deadline \( d_i \), best-case execution time (BCET) \( C_{i_{\min}} \), WCET \( C_{i_{\max}} \), and priority \( p_i \). The priority of a job can be decided by the system designer at design time or by the system’s JLFP scheduling algorithm. We assume that a numerically smaller value of \( p_i \) implies higher priority. Any ties in priority are broken by job ID. For ease of notation, we assume that the “<” operator implicitly reflects this tie-breaking rule. We use \( \mathbb{N} \) to represent the natural numbers including 0. We assume a discrete-time model and all job timing parameters are in \( \mathbb{N} \).
At runtime, each job is released at an a priori unknown time \( r_i \in [r_{i\min}, r_{i\max}] \). We say that a job \( J_i \) is possibly released at time \( t \) if \( t \geq r_{i\min} \), and certainly released if \( t \geq r_{i\max} \)

Such release jitter may arise due to timer inaccuracy, interrupt latency, or communication delays, e.g., when the task is activated after receiving data from the network. Similarly, each released job has an a priori unknown execution time requirement \( C_i \in [C_{i\min}, C_{i\max}] \).

Execution time variation occurs because of the use of caches, out-of-order-execution, input dependencies, program path diversity, state dependencies, etc. We assume that the absolute deadline of a job, i.e., \( d_i \), is fixed a priori and not affected by release jitter. Released jobs remain pending until completed, i.e., there is no job-discarding policy.

Each job must execute sequentially, i.e., it cannot execute on more than one core at a time. Hence, because jobs are non-preemptive, a job \( J_i \) that starts its execution on a core at time \( t \) occupies that core during the interval \([t, t + C_i]\). In this case, we say that job \( J_i \) finishes by time \( t + C_i \). At time \( t + C_i \), the core used by \( J_i \) becomes available to start executing other jobs. A job’s response time is defined as the length of the interval between the arrival and completion of the job [2], i.e., \( t + C_i - r_{i\min} \). We say that a job is ready at time \( t \) if it is released and did not yet start its execution prior to time \( t \).

In this paper, we assume that shared resources that must be accessed in mutual exclusion are protected by FIFO spin locks. Since we consider a non-preemptive execution model, it is easy to obtain a bound on the worst-case time that any job spends spinning while waiting to acquire a contented lock; we assume the worst-case spinning delay is included in the WCETs.

Throughout the paper, we use \( \{\cdot\} \) to denote a set of items in which the order of elements is irrelevant and \( \langle\cdot\rangle \) to denote an enumerated set of items. In the latter case, we assume that items are indexed in the order of their appearance in the sequence. For ease of notation, we use \( \max_0\{X\} \) and \( \min_\infty\{X\} \) over a set of positive values \( X \subseteq \mathbb{N} \) that is completed by 0 and \( \infty \), respectively. That is, if \( X = \emptyset \), then \( \max_0\{X\} = 0 \) and \( \min_\infty\{X\} = \infty \), otherwise they return the usual maximum and minimum values in \( X \), respectively.

The schedulability analysis proposed in this paper can be applied to periodic tasks. A thorough discussion of how many jobs must be considered in the analysis for different types of tasks with release offset and constrained or arbitrary deadlines has been presented in [16].

**Scheduler model.** We consider a non-preemptive global JLFP scheduler upon an identical multicore platform. The scheduler is invoked whenever a job is released or completed. In the interest of simplifying the presentation of the proposed analysis, we make the modeling assumption that, without loss of generality, at any invocation of the scheduler, at most one job is picked and assigned to a core. If two or more release or completion events occur at the same time, the scheduler is invoked once for each event. The actual scheduler implementation in the analyzed system need not adhere to this restriction and may process more than one event during a single invocation. Our analysis remains safe if the assumption is relaxed in this manner.

We allow for a non-deterministic core-selection policy when more than one core is available for executing a job, i.e., when a job is scheduled, it may be scheduled on any available core. The reason is that requiring a deterministic tie-breaker for core assignments would impose a large synchronization overhead, e.g., to rule out any race windows when the scheduler is invoked concurrently on different cores at virtually the same time, and hence no such rule is usually implemented in operating systems.

We say that a job set \( J \) is schedulable under a given scheduling policy if no execution scenario of \( J \) results in a deadline miss, where an execution scenario is defined as follows.

**Definition 1.** An execution scenario \( \gamma = \{(r_1, C_1), (r_2, C_2), \ldots, (r_n, C_n)\} \), where \( n = |J| \), is an assignment of execution times and release times to the jobs of \( J \) such that, for each
job $J_i$, $C_i \in [C_{i}^{\min}, C_{i}^{\max}]$ and $r_i \in [r_{i}^{\min}, r_{i}^{\max}]$.

We exclusively focus on work-conserving, and priority-driven scheduling algorithms, i.e., the scheduler dispatches a job only if the job has the highest priority among all ready jobs, and it does not leave a core idle if there exists a ready job. We assume that the WCET of each job is padded to cover the scheduler overhead and to account for any micro-architectural interference (e.g., cache or memory bus interference).

3 Schedule-Abstraction Graph

Our schedulability analysis derives a safe upper bound on the WCRT and a safe lower bound on the BCRT of each job by exploring a superset of all possible schedules. Since the number of schedules depends on the space of possible execution scenarios, which is a combination of release times and execution times of the jobs, it is intractable to naively enumerate all distinct schedules. To solve this problem, we aggregate schedules that lead to the same ordering of job start times (a.k.a. dispatch times) on the processing platforms. To this end, in the rest of this section, we introduce an abstraction of job orderings that encodes possible finish times of the jobs.

To represent possible job orderings we use an acyclic graph whose edges are labeled with jobs. Thus, each path in the graph represents a dispatch order of jobs in the system. Fig. 1-(b) shows an example of such a graph. For example, the path from $v_1$ to $v_9$ means that the jobs $(J_1, J_2, J_3, J_4, J_5)$ have been scheduled one after another. The length of a path $P$, denoted by $|P|$, is the number of jobs scheduled on that path.

To account for the uncertainties in the release times and execution times of jobs, which in turn result in different schedules, we use intervals to represent the state of a core. For example, assume that there is only one core in the system and consider a particular job $J_i$. Assume that the release interval and execution requirement of $J_i$ are $[0, 5]$ and $[10, 15]$, respectively. In a job ordering where $J_i$ is the first job dispatched on the core, the resulting core interval will become $[10, 20]$, where $10 = r_{i}^{\min} + C_{i}^{\min}$ and $20 = r_{i}^{\max} + C_{i}^{\max}$ are the earliest finish time (EFT) and latest finish time (LFT), respectively, of the job on the core. Here, the interval $[10, 20]$ means that the core will be possibly available at time $10$ and will be certainly available at time $20$. Equivalently, any time instant $t$ in a core interval corresponds to an execution scenario in which the core is busy until $t$ and becomes available at $t$.

Using the notion of core intervals, we define a system state as a set of $m$ core intervals. System states are vertices of the graph and represent the states of the cores after a certain set of jobs has been scheduled in a given order.

3.1 Graph Definition

The schedule-abstraction graph is a directed acyclic graph $G = (V, E)$, where $V$ is a set of system states and $E$ is the set of labeled edges. A system state $v \in V$ is a multiset of $m$ core intervals denoted by $\{\phi_1, \phi_2, \ldots, \phi_m\}$. A core interval $\phi_k = [EFT_k, LFT_k]$ is defined by the EFT and LFT of a job that is scheduled on the core, denoted by $EFT_k$ and $LFT_k$, respectively. Equivalently, $EFT_k$ is the time at which the core becomes possibly available and $LFT_k$ is the time at which the core becomes certainly available. Since cores are identical, the schedule-abstraction graph does not distinguish between them and hence does not keep track of the physical core on which a job is executing.

The schedule-abstraction graph contains all possible orderings of job start times in any possible schedule. This ordering is represented by directed edges. Each edge $e = (v_p, v_q)$
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from state $v_p$ to state $v_q$ has a label representing the job that is scheduled next after state $v_p$. The sequence of edges in a path $P$ represents a possible sequence of scheduling decisions (i.e., a possible sequence of job start times) to reach the system state modeled by $v_p$ from the initial state $v_1$.

### 3.2 Example

Fig. 1-(b) shows the schedule-abstraction graph that includes all possible start-time orders of the jobs defined in Fig. 1-(a) on a two-core processor. In the initial state $v_1$, no job is scheduled. At time 0, two jobs $J_1$ and $J_2$ are released. Since $p_1 < p_2$, the scheduler first schedules $J_1$ on one of the available cores. For the sake of clarity, we have numbered the cores in this example, however, they are identical from our model’s perspective.

Fig. 1-(c) shows the state of both cores after job $J_1$ is scheduled. The dashed rectangle that covers the interval $[0, 2)$ shows the time during which the core is certainly available from time 4. Thus, when job $J_3$ is released at time 5, the scheduler has no other choice but to schedule job $J_3$ on this core. The label of this transition shows that $J_3$ has been scheduled.

From system state $v_4$, two other states are reachable depending on the finish times of jobs $J_2$ and $J_3$. State $v_5$: If core $\phi_1$ becomes available before core $\phi_2$, then $J_1$ can start its execution on $\phi_1$. This results in state $v_6$ (Fig. 1-(f)). The core intervals of $v_6$ are obtained

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**Figure 1** A schedule-abstraction graph $G$ for five jobs that are scheduled on two cores: (a) shows the job set information (jobs do not have release jitter), (b) shows the schedule-abstraction graph, (c) to (k) show the state of the two cores at system states $v_2$ to $v_{10}$, respectively.
as follows. According to the intervals of \( v_4 \), the earliest time at which \( \phi_1 \) becomes available is 6, while the release time of \( J_4 \) is 8, thus, the earliest start time of \( J_4 \) on core \( \phi_1 \) is 8, which means that its earliest finish time is 10. The latest start time of \( J_4 \) such that it is still scheduled on core \( \phi_1 \) is time 12. The reason is that \( J_4 \) is released at time 8 and hence is pending from that time onward. However, it cannot be scheduled until a core becomes available. The earliest time a core among \( \phi_1 \) and \( \phi_2 \) becomes available is at time 12 (which is the latest finish time of \( J_3 \)). Since the scheduling algorithm is work-conserving, it will certainly schedule job \( J_4 \) at 12 on the core that has become available. Consequently, the latest finish time of \( J_4 \) is 12 + 3 = 15.

**State** \( v_6 \): In state \( v_4 \), if core \( \phi_2 \) becomes available before \( \phi_1 \), then job \( J_4 \) can be scheduled on \( \phi_2 \) and create state \( v_6 \) (Fig. 1-(g)). In this case, the earliest start time of \( J_4 \) is at time 10 because, although it has been released before, it must wait until core \( \phi_2 \) becomes available, which happens only at time 10. As a result, the earliest finish time of \( J_4 \) will be time 10 + 2 = 12. On the other hand, the latest start time of \( J_4 \) such that it is scheduled on core \( \phi_2 \) is 12 because at this time, job \( J_4 \) is ready and a core (\( \phi_1 \)) becomes available. Thus, if \( J_4 \) is going to be scheduled on \( \phi_2 \), core \( \phi_2 \) must become available by time 12. Note that since our core-selection policy is non-deterministic, if \( \phi_2 \) becomes available at time 12, \( J_4 \) may be dispatched on either core. Consequently, the latest finish time of \( J_4 \) when scheduled on \( \phi_2 \) is 12 + 3 = 15. Furthermore, system state \( v_6 \) may arise only if core \( \phi_1 \) has not become available before time 10, as otherwise job \( J_4 \) will be scheduled on \( \phi_1 \) and create state \( v_5 \). Thus, state \( v_6 \) can be reached only if \( \phi_1 \) does not become available before time 10. To reflect this constraint, the core interval of \( \phi_1 \) must be updated to [10, 12]. The red dashed rectangle in Fig. 1-(g) illustrates this update. According to the schedule-abstraction graph in Fig. 1-(b), there exist three scenarios in which \( J_5 \) finishes at time 16 and hence misses its deadline. These scenarios are shown in Figs. 1-(h), (i) and (k), and are reflected in states \( v_7, v_8, \) and \( v_{10} \), respectively.

## 4 Schedulability Analysis

This section explains how to build the schedule-abstraction graph. Sec. 4.1 presents the high-level description of our search algorithm, which consists of alternating expansion, fast-forward, and merge phases. These phases will be discussed in details in Sec. 4.2, 4.3, and 4.4, respectively. Sec. 5 provides a proof of correctness of the proposed algorithm.

### 4.1 Graph-Generation Algorithm

During the expansion phase, (one of) the shortest path(s) \( P \) in the graph from the root to a leaf vertex \( v_p \) is expanded by considering all jobs that can possibly be chosen by the JLFP scheduler to be executed next in the job execution sequence represented by \( P \). For each such job, the algorithm checks on which core(s) it may execute. Finally, for each core on which the job may execute, a new vertex \( v'_p \) is created and added to the graph, and connected via an edge directed from \( v_p \) to \( v'_p \).

After generating a new vertex \( v'_p \), the fast-forward phase advances time until the next scheduling event. It accordingly updates the system state represented by \( v'_p \).

The merge phase attempts to moderate the growth of the graph. To this end, the terminal vertices of paths that have the same set of scheduled jobs (but not necessarily in the same order) and core states that will lead to similar future scheduling decisions by the scheduler, are merged into a single state whose future states cover the set of all future states of the merged states. The fast-forward and merge phases are essential to avoid redundant work, i.e., to recognize that two or more states are similar early on before they are expanded. The
Algorithm 1: Schedule Graph Construction Algorithm

**Input:** Job set $J$

**Output:** Schedule graph $G = (V, E)$

1. $\forall j_i \in J$, $BCRT_i \leftarrow \infty$, $WCRT_i \leftarrow 0$;
2. Initialize $G$ by adding a root vertex $v_1 = \{[0, 0], [0, 0], \ldots, [0, 0]\}$, where $|v_1| = m$;
3. while $\exists$ a path $P$ from $v_1$ to a leaf vertex $v_p$ s.t. $|P| < |J|$ do
   4. $P \leftarrow$ a path from $v_1$ to a leaf with the least number of edges in the graph;
   5. $v_p \leftarrow$ the leaf vertex of $P$;
   6. for each job $j_i \in J \setminus J^P$ do
      7. for each core $\phi_k \in v_p$ do
         8. if $j_i$ can be dispatched on core $\phi_k$ according to (1) then
            9. Build $v'_p$ using (10);
            10. $BCRT_i \leftarrow \min\{EFT'_{k} - r_{i}^{\min}, BCRT_i\}$;
            11. $WCRT_i \leftarrow \max\{LFT'_{k} - r_{i}^{\min}, WCRT_i\}$;
            12. Connect $v_p$ to $v'_p$ by an edge with label $j_i$;
            13. Fast-forward $v'_p$ according to (13);
            14. while $\exists$ path $Q$ that ends to $v_q$ such that the condition defined in Definition 2 is satisfied for $v'_p$ and $v_q$ do
               15. Update $v'_p$ using Algorithm 2;
               16. Redirect all incoming edges of $v_q$ to $v'_p$;
               17. Remove $v_q$ from $V$;
            end
        end
   end
4. end
5. end
6. end
7. end
8. end
9. end

The algorithm terminates when there is no vertex left to expand, that is, when all paths in the graph represent a valid schedule of all jobs in $J$.

Algorithm 1 presents our iterative method to generate the schedule- abstraction graph in full detail. A set of variables keeping track of a lower bound on the BCRT and an upper bound on the WCRT of each job is initialized at line 1. These bounds are updated whenever a job $j_i$ can possibly be scheduled on any of the cores. The graph is initialized at line 2 with a root vertex $v_1$. The expansion phase corresponds to lines 6–21; line 13 implements the fast-forward, and lines 14–18 realize the merge phase. These phases repeat until every path in the graph contains $|J|$ distinct jobs. We next discuss each phase in detail.

### 4.2 Expansion Phase

Assume that $P$ is a path connecting the initial state $v_1$ to $v_p$. The sequence of edges in $P$ represents a sequence of scheduling decisions (i.e., a possible sequence of job executions) to reach the system state modeled by $v_p$ from the initial state $v_1$. We denote by $J^P$ the set of jobs scheduled in path $P$. To expand path $P$, Algorithm 1 evaluates for each job $j_i \in J \setminus J^P$ that was not scheduled yet whether it may be the next job picked by the scheduler and scheduled on any of the cores. For any job $j_i$ that can possibly be scheduled on a core $\phi_k \in v_p$ before any other job starts executing, a new vertex $v'_p$ is added to the graph (see lines 6–12 of Algorithm 1).
To evaluate if job $J_i$ is a potential candidate for being started next in the dispatch sequence represented by $P$, we need to know:

1. The earliest time at which $J_i$ may start to execute on core $\phi_k$ when the system is in the state described by vertex $v_p$. We call that instant the *earliest start time* (EST) of $J_i$ on core $\phi_k$, and we denote it by $EST_{i,k}(v_p)$.

2. The time by which $J_i$ must have certainly started executing if it is to be the *next job* to be scheduled by the JLFP scheduler on the processing platform. This second time instant is referred to as the *latest start time* (LST) of $J_i$ and is denoted by $LST_{i}(v_p)$.

$LST_{i}(v_p)$ represents the latest time at which a work-conserving JLFP scheduler schedules $J_i$ next after state $v_p$. Note that $LST_{i}(v_p)$ is a global value for the platform when it is in state $v_p$, while $EST_{i,k}(v_p)$ is related to a specific core $\phi_k$.

A job $J_i$ can be the next job scheduled in the job sequence represented by $P$ if there is a core $\phi_k$ for which the earliest start time $EST_{i,k}(v_p)$ of $J_i$ on $\phi_k$ is not later than the latest time at which this job must have started executing, i.e., before $LST_{i}(v_p)$ (see Lemma 2 in Sec. 5 for a formal proof). That is, $J_i$ may commence execution on $\phi_k$ only if

$$EST_{i,k}(v_p) \leq LST_{i}(v_p).$$

(1)

For each core $\phi_k$ that satisfies (1), a new vertex $v'_p$ is created, where $v'_p$ represents the state of the system after dispatching job $J_i$ on core $\phi_k$.

Below, we explain how to compute $EST_{i,k}(v_p)$ and $LST_{i}(v_p)$. Then we describe how to build a new vertex $v'_p$ for each core $\phi_k$ and job $J_i$ that satisfies (1). Finally, we explain how the BCRT and WCRT of job $J_i$ are updated according to its $EST_{i,k}(v_p)$ and $LST_{i}(v_p)$, respectively. To ease readability, from here on we will not specify any more that $\phi_k$, $EST_{i,k}(v_p)$ and $LST_{i}(v_p)$ are related to a specific vertex $v_p$ when it is clear from context, and will instead use the short-hand notations $EST_{i,k}$ and $LST_{i}$.

**Earliest start time.** To start executing on a core $\phi_k$, a job $J_i$ has to be released and $\phi_k$ has to be available. Thus, the earliest start time $EST_{i,k}$ of a job $J_i$ on a core $\phi_k$ is given by

$$EST_{i,k} = \max\{r^\text{min}_i, EFT_k\}.$$  

(2)

where $r^\text{min}_i$ is the earliest time at which $J_i$ may be released and $EFT_k$ is the earliest time at which $\phi_k$ may become available.

**Latest start time.** Because we assume a work-conserving JLFP scheduling algorithm, two conditions must hold for job $J_i$ be the next job scheduled on the processing platform: (i) $J_i$ must be the highest-priority ready job (because of the JLFP assumption), and (ii) for every job $J_j$ released before $J_i$, either $J_j$ was already scheduled earlier on path $P$ (i.e., $J_j \in J^P$), or all cores were busy from the release of $J_j$ until the release of $J_i$.

If (i) is not satisfied, then a higher-priority ready job is scheduled instead of $J_i$. Therefore the latest start time $LST_i$ of $J_i$ must be earlier than the earliest time at which a not-yet-scheduled higher-priority job is certain to be released, that is, $LST_i < t_{\text{high}}$, where

$$t_{\text{high}} = \min_{x} \{ r_x^{\text{max}} \mid x \in J \setminus J^P \land p_x < p_i \}.$$  

(3)

If (ii) is not satisfied, then an earlier released job $J_j$ will start executing on an idle core before $J_i$ is released. Therefore the latest start time $LST_i$ of $J_i$ cannot be later than the earliest time at which both a core is certainly idle and a not-yet-scheduled job is certainly
released. Formally, \( LST_i \leq t_{uc} \), where
\[
\begin{align*}
t_{uc} & \triangleq \max\{t_{core}, t_{job}\}, \\
t_{core} & \triangleq \min\{LF_{t_x} \mid 1 \leq x \leq m\}, \quad \text{and} \\
t_{job} & \triangleq \min\{r^m_{y_{max}} \mid J_y \in \mathcal{J} \setminus \mathcal{J}^P\}.
\end{align*}
\]
In the equations above, \( t_{core} \) is the earliest time at which a core is certainly idle and \( t_{job} \) is the earliest time at which a not-yet-scheduled job is certainly released.

Combining \( LST_i \leq t_{high} \) and \( LST_i \leq t_{uc} \), we observe that \( J_i \) must start by time
\[
LST_i = \min\{t_{uc}, t_{high} - 1\}.
\]

\textbf{Example 1.} Fig. 2-(a) shows how \( EST_{i,k} \) and \( LST_i \) are calculated when job \( J_i \) is scheduled on core \( \phi_2 \). In this example, \( t_{job} = 14 \) since job \( J_x \) becomes certainly available at that time. However, the earliest time at which a core (in this case, core \( \phi_1 \)) becomes available is \( t_{core} = 24 \), thus, \( t_{uc} = 24 \). On the other hand, the earliest time at which a job with a higher priority than \( J_i \) is certainly released is \( t_{high} = 17 \). Thus, \( LST_i = t_{high} - 1 = 16 \).

\textbf{Building a new system state.} If Inequality (1) holds, it is possible that job \( J_i \) is the next successor of path \( P \) and is scheduled on core \( \phi_k \) at any \( t \in [EST_{i,k}, LST_i] \) (Lemma 2 in Sec. 5 proves this claim). Our goal is to generate a single new vertex for the schedule-abstraction graph that aggregates all these execution scenarios.

Let \( v'_p \) denote the vertex that represents the new system state resulting from the execution of job \( J_i \) on core \( \phi_k \). The earliest and latest times at which \( \phi_k \) may become available after executing job \( J_i \) is obtained as follows:
\[
EFT'_{k} = EST_{i,k} + C^i_{min} \quad \text{and} \quad LFT'_{k} = LST_i + C^i_{max}.
\]
Furthermore, because the latest scheduling event in the system state \( v'_p \) occurs no earlier than \( EST_{i,k} \), no other job in \( \mathcal{J} \setminus \mathcal{J}^P \) may possibly be scheduled before \( EST_{i,k} \).

\textbf{Property 1.} If job \( J_i \) is the next job scheduled on the platform, and if it is scheduled on core \( \phi_k \), then no job in \( \mathcal{J} \setminus \mathcal{J}^P \) starts executing on any core \( \phi_x, 1 \leq x \leq m \) before \( EST_{i,k} \).

\textbf{Proof.} By contradiction. Assume a job \( J_j \in \mathcal{J} \setminus \mathcal{J}^P \) starts executing on a core \( \phi_x \) before \( EST_{i,k} \). Because \( J_i \) cannot start executing on \( \phi_k \) before \( EST_{i,k} \), \( J_j \) must be different from \( J_i \) and hence \( J_j \) starts to execute before \( J_i \). That contradicts the assumption that \( J_i \) is the first job in \( \mathcal{J} \setminus \mathcal{J}^P \) to be scheduled on the platform.
To ensure that Property 1 is correctly enforced in the new system state represented by \( v_p' \), we update the core intervals in state \( v_p' \) as follows

\[
\phi'_x \triangleq \begin{cases} 
[ EFT'_k, LFT'_k ] & \text{if } x = k, \\
[ EST_{i,k}, EST_{i,k} ] & \text{if } x \neq k \land LFT_x \leq EST_{i,k}, \\
\{ \max\{ EST_{i,k}, EFT_x \}, LFT_x \} & \text{otherwise.}
\end{cases}
\]  

(9)

The first case of (9) simply repeats (8) for job \( J_i \). The second and third cases ensure that no job in \( J \setminus \mathcal{J}_P \) can be scheduled on those cores before \( EST_{i,k} \). This is done by forcing \( \phi_x \)'s earliest availability time to be equal to \( EST_{i,k} \). Finally, for cores that would certainly be idle after \( EST_{i,k} \) (i.e., the second case in (9)), we set \( LFT_k \) (i.e., the time at which it becomes certainly available) to \( EST_{i,k} \).

Finally, the new vertex \( v_p' \) is generated by applying (9) on all cores, i.e.,

\[
v_p' = \{ \phi'_1, \phi'_2, \ldots, \phi'_m \}. \]  

(10)

### Deriving the BCRT and WCRT of the jobs

Recall that the BCRT and the WCRT of a job are relative to its arrival time, i.e., \( r^\text{min}_i \), and not its actual release time, which can be any time between \( r^\text{min}_i \) and \( r^\text{max}_i \). In other words, release jitter counts towards a job’s response time. As stated earlier, the earliest finish time of \( J_i \) on core \( \phi_k \) cannot be smaller than \( EFT'_k \) and the latest finish time of \( J_i \) on core \( \phi_k \) cannot be larger than \( LFT'_k \) (obtained from (8)). Using these two values, the BCRT and WCRT of job \( J_i \) are updated at lines 10 and 11 of Algorithm 1 as follows.

\[
\text{BCRT}_i \leftarrow \min\{EFT'_k - r^\text{min}_i, \text{BCRT}_i\} \quad (11)
\]

\[
\text{WCRT}_i \leftarrow \max\{LFT'_k - r^\text{min}_i, \text{WCRT}_i\} \quad (12)
\]

If the algorithm terminates, then \( \text{WCRT}_i \) and \( \text{BCRT}_i \) contain an upper bound on the WCRT and a lower bound on the BCRT of job \( J_i \), respectively, over all paths. Since the graph considers all possible execution scenarios of \( J \), it considers all possible schedules of \( J_i \). The resulting WCRT and BCRT estimates are therefore safe bounds on the actual WCRT and BCRT of the job, respectively. This property is proven in Corollary 3 in Sec. 5.

The quality of service of many real-time systems depends on both the WCRT and response-time jitter [7] of each task, i.e., the difference between the BCRT and WCRT of that task. One of the advantages of our schedule-abstraction graph is that it not only provides a way to compute those quantities, but also allows to extract the maximum variation between the response times of successive jobs released by the same task, hence allowing a more accurate analysis of (for instance) sampling jitter in control systems.

### 4.3 Fast-Forward Phase

As shown in lines 6 and 7, one new state will be added to the graph for each not-yet-scheduled job that can be scheduled next on one of the cores. This situation can lead to an explosion in the search space if the number of states is not reduced. In this work, we merge states to avoid redundant future explorations. To aid the subsequent merge phase, the fast-forward phase advances the time until a job may be released. We denote that instant by \( t_{\text{min}} \triangleq \min\{ r^\text{min}_x \mid J_x \in J \setminus \mathcal{J}_P \setminus \{J_i\} \} \). The fast-forward phase thus updates each core interval \( \phi'_x \in v_p' \) as follows:

\[
\phi'_x = \begin{cases} 
[t_{\text{min}}, \max\{ t_{\text{min}}, EFT_x \}, LFT_x ] & \text{if } LFT_x \leq t_{\text{min}}, \\
[t_{\text{min}}, \max\{ t_{\text{min}}, EFT_x \}, LFT_x ] & \text{otherwise.}
\end{cases}
\]  

(13)
**Algorithm 2:** Algorithm that merges $v_p$ and $v_q$, and creates $v'_p$.

1. Sort and re-index the core intervals $\phi_x(v_p)$ of $v_p$ in a non-decreasing order of their EFTs, such that $EFT_1(v_p) \leq EFT_2(v_p) \leq \ldots \leq EFT_m(v_p)$;
2. Sort and re-index $v_q$'s core intervals in a non-decreasing order of their EFTs such that $EFT_1(v_q) \leq EFT_2(v_q) \leq \ldots \leq EFT_m(v_q)$;
3. Pair each two core intervals $\phi_x(v_p)$ and $\phi_y(v_q)$ to create $\phi_x(v'_p) \triangleq \min\{EFT_x(v_p), EFT_x(v_q)\}, \max\{LFT_x(v_p), LFT_x(v_q)\}$;

The first case of (13) relies on the fact that from $LFT'_x$ onward (i.e., the time at which a core $\phi'_x$ becomes certainly available), $\phi'_x$ remains available until a new job is scheduled on it. Since the earliest time at which a job can be scheduled is $t_{min}$, this core remains available at least until $t_{min}$. Thus, it is safe to update its interval to $[t_{min}, t_{min}]$, which denotes that the core is certainly free by $t_{min}$. Similarly, the second case of (13) is based on the fact that a core $\phi_x$ that is possibly available at $EFT'_x$ remains possibly available either until reaching $LFT'_x$ (where it certainly becomes free) or until a job may be scheduled on $\phi_x$, which does not happen until $t_{min}$ at the earliest. Lemma 4 in Sec. 5 proves that fast-forwarding state $v'_p$ will not change any of the future states that can be reached from $v'_p$ before applying (13).

### 4.4 Merge Phase

The merge phase seeks to collapse states to avoid redundant future explorations. The goal is to reduce the size of the search space such that the computed BCRT of any job may never become larger, the computed WCRT of any job may never become smaller, and all job scheduling sequences that were possible before merging states are still considered after merging those states. The merge phase is implemented in lines 14–18 of Algorithm 1, where the condition defined below in Definition 2 is evaluated for paths with length $|P| + 1$.

Since each state consists of exactly $m$ core intervals, merging two states requires finding a matching among the two sets of intervals to merge individual intervals. Let states $v_p$ and $v_q$ be the end vertices of two paths $P$ and $Q$. In order to merge $v_p$ and $v_q$ into a new state $v'_p$, we apply Algorithm 2. Next, we establish our merging rules, which will be proven to be safe in Corollary 1 in Sec. 5.

**Definition 2.** Two states $v_p$ and $v_q$ can be merged if (i) $J^P = J^Q$, (ii) $\forall \phi_i(v_p), \phi_i(v_q)$, $\max\{EFT_i(v_p), EFT_i(v_q)\} \leq \min\{LFT_i(v_p), LFT_i(v_q)\}$, and (iii) at any time $t$, the number of possibly-available cores in the merged state must be equal to the number of possibly-available cores in $v_p$ or $v_q$, i.e.,

$$\forall t \in T, B(t, v'_p) = B(t, v_p) \lor B(t, v'_q) = B(t, v_q),$$

(14)

where $B(t, v_x)$ counts the number of core intervals of a state $v_x$ that contain $t$, i.e.,

$$B(t, v_x) = \left| \{ \phi_y(v_z) \mid t \in [EFT_y(v_z), LFT_y(v_z)] \} \right|,$$

(15)

and where $T$ is the set of time instants at which the value of $B(\cdot)$ may change, i.e.,

$$T = \{ EFT_x(v_p) \mid \forall x \} \cup \{ LFT_x(v_p) \mid \forall x \} \cup \{ EFT_x(v_q) \mid \forall x \} \cup \{ LFT_x(v_q) \mid \forall x \}.$$

(16)

**Example 2.** Fig. 2-(b) shows two states $v_p$ and $v_q$ that are merged to create state $v'_p$. As shown, for any $t \in T$, $B(t, v'_p)$ is equal to $B(t, v_p)$ or $B(t, v_q)$.
Notably, any merge rule that respects condition (i) in Definition 2 is safe (see Corollary 1 in Sec. 5.3). The role of conditions (ii) and (iii) is to trade-off between the accuracy and performance of the analysis by evading the inclusion of impossible execution scenarios in the resulting state. We leave the investigation of more accurate (or more eager) merging conditions, as well as the applicability of abstraction-refinement techniques, to future work.

5 Correctness of the Proposed Solution

In this section, we show that the schedule-abstraction graph constructed by Algorithm 1 correctly includes all job schedules that can arise from any possible execution scenario, i.e., for any possible execution scenario, there exists a path in the graph that represents the schedule of those jobs in that execution scenario (Theorem 1). The proof has two main steps: we first assume that the fast-forward and merge phases are not executed and show that the EFT and LFT of a job obtained from Equation (8) are correct lower and upper bounds on the schedule of those jobs in that execution scenario (Lemma 3). Finally, we establish that Algorithm 1 correctly derives an upper bound on the WCRT and a lower bound on the BCRT of every job (Corollary 3).

5.1 Soundness of the Expansion Phase

In this section, we assume that neither the fast-forward nor the merge phase is executed.

\begin{lemma}
For any vertex \( v_p \) and any successor \( v'_p \) of \( v_p \), such that job \( J_i \in J \setminus J^P \) is scheduled on core \( \phi_k \) between \( v_p \) and \( v'_p \), EFT\(_k\)(\( v'_p \)) and LFT\(_k\)(\( v'_p \)) (as computed by (8)) are safe bounds on the finish time of \( J_i \).
\end{lemma}

\textbf{Proof.} If neither the fast-forward nor the merge phases are executed, (9) is the only equation used to build a new state \( v'_p \). In this lemma, we first prove that the EST and LST of the job obtained from (2) and (7) are a lower and an upper bound on the start time of job \( J_i \) on \( \phi_k \) after the scheduling sequence represented by \( P \). Then, we conclude that EFT\(_k\)(\( v'_p \)) and LFT\(_k\)(\( v'_p \)) are safe bounds on the finish time of \( J_i \) on \( \phi_k \). The proof is by induction.

\textbf{Base case.} The base case is for any vertex \( v'_p \) that succeeds to the root vertex \( v_1 \) where all cores are idle. Hence in \( v'_p \), job \( J_i \) is scheduled on one of the idle cores, say \( \phi_k \). Since all cores are idle at time 0, Equation (2) yields \( EST_{i,k}(v_1) = r^\text{min}_i \), which is by definition the earliest time at which job \( J_i \) may start. Consequently, the earliest finish time of \( J_i \) cannot be smaller than \( EFT_k(v'_p) = r^\text{min}_i + C^\text{min}_i \).

Similarly, (7) yields \( LST_i(v_1) = \min\{t_{\text{high}} - 1, t_{\text{job}}\} \) (recall that \( t_{\text{core}} = 0 \) since all cores are idle in \( v_1 \)). \( J_i \) cannot start later than \( LST_i(v_1) = t_{\text{job}} \) if it is the first scheduled job as all cores are idle and hence as soon as a job is certainly released, it will be scheduled right away on one of the idle cores. Similarly, \( J_i \) cannot start its execution if it is not the highest-priority job anymore, i.e., at or after time \( t_{\text{high}} \). As a result, the latest finish time of \( J_i \) cannot be larger than \( LFT_k(v'_p) = \min\{t_{\text{job}}, t_{\text{high}} - 1\} + C^\text{max}_i \). Therefore, EFT\(_k\)(\( v'_p \)) and LFT\(_k\)(\( v'_p \)) are safe bounds on the finishing time of \( J_i \) on \( \phi_k \) after the scheduling sequence \( P = (v_1, v'_p) \).
Lemma 2. If job $J_i$ is the first job executing on the platform and $J_i$ is not released before $r_{i}^{\min}$.

**Induction step.** Assume now that each core interval on every vertex from $v_1$ to $v_p$ along path $P$ provides a lower bound and an upper bound on the time at which that core will possibly and certainly be available, respectively, to start executing a new job. We show that in the new vertex $v_p'$ obtained from scheduling job $J_i$ on core $\phi_k$ after $P$, (8) provides a safe lower and upper bound on the finish time of $J_i,$ and for other cores, the new core intervals computed by (9) are safe, i.e., no new job can start its execution on a core $\phi_x$ before $EFT_x$ and the core cannot remain busy after $LFT_x$.

**EFT.** The earliest start time of $J_i$ on core $\phi_k$, i.e., $EST_{i,k}(v_p') = EFT_{k}(v_p') = r_{i,k}^{\min}$ (recall that $EFT_{k}(v_1) = LFT_{k}(v_1) = 0$), which is indeed the earliest time at which any job may start on $\phi_x$ if $J_i$ is the first job executing on the platform and $J_i$ is not released before $r_{i}^{\min}$.

**LFT.** Next, we show that $LST_i(v_p')$ cannot exceed $t_{high} - 1$ or $t_{uc}$ as stated by (7). First, consider $t_{high}$ and suppose $t_{high} \neq \infty$ (otherwise the claim is trivial). Since a higher-priority job is certainly released at the latest at time $t_{high}$, job $J_i$ is no longer the highest-priority job at time $t_{high}$. Consequently, it cannot commence execution under a JLF scheduler at or after time $t_{high}$ if it is to be the next job scheduled after $P$. Hence, job $J_i$ will be a direct successor of path $P$ only if its execution starts no later than time $t_{high} - 1$. Now, consider $t_{uc}$. At time $t_{uc}$, a not-yet-scheduled job is certainly released and a core is certainly available. Hence a work-conserving scheduler will schedule that job at $t_{uc}$, thus, job $J_i$ will be a direct successor of path $P$ only if its execution starts no later than time $t_{uc}$. Since $LST_i(v_p')$ is the upper bound on the time at which job $J_i$ can start its execution while being the next job scheduled after path $P$, the latest finish time of $J_i$ on core $\phi_k$ cannot be larger than $\min\{t_{high} - 1, t_{uc}\} + C_{i,k}^{max}$, which proves the correctness of (8) for $LFT_{k}(v_p')$.

Since in state $v_p'$ job $J_i$ is scheduled on core $\phi_k$ other cores cannot be available before $EST_{i,k}$, otherwise a work-conserving scheduler would schedule $J_i$ on one of those cores instead of on $\phi_k$. Equation (9) ensures that if $J_i$ is the next job to be scheduled and if $\phi_k$ is the core on which $J_i$ is scheduled, no other core will certainly be available by $EST_{i,k}(v_p')$, i.e., $EFT_{k}(v_p') \geq EST_{i,k}(v_p')$.

By induction on all vertices in $V$, we have that $EFT_{k}(v_p')$ is a safe bound on the finish time of any job scheduled between any two states $v_p$ and $v_p'$, including $J_i$. ◀

**Lemma 2.** Job $J_i$ can be scheduled next on core $\phi_k$ after jobs in path $P$ only if (1) holds.

**Proof.** If job $J_i$ is released at time $r_{i,k}^{\min}$ and the core $\phi_k$ becomes available at $EFT_{k}$, then it can be dispatched no earlier than at time $EST_{i,k} = \max\{r_{i,k}^{\min}, EFT_{k}\}$. If (1) does not hold, then $t_{high}$ or $t_{uc}$ (or both) are smaller than $EST_{i,k}$. This implies that either a higher-priority job other than job $J_i$ is certainly released before $EST_{i,k}$ or a job other than $J_i$ is certainly
released before $EST_{i,k}$ and a core is certainly available before $EST_{i,k}$. In both cases, a work-conserving JLFP scheduling algorithm will not schedule job $J_i$ until that other job is scheduled. Consequently, job $J_i$ cannot be the next successor of path $P$.

**Lemma 3.** Assuming that neither the fast-forward nor the merge phases are executed in Algorithm 1, for any execution scenario such that a job $J_i \in \mathcal{J}$ completes at some time $t$ on core $\phi_k$ (under the given scheduler), there exists a path $P = \langle v_1, \ldots, v_p, v_p' \rangle$ in the schedule-abstraction graph such that $J_i$ is the label of the edge from $v_p$ to $v_p'$ and $t \in [EFT_k(v_p'), LFT_k(v_p')]$, where $EFT_k(v_p')$ and $LFT_k(v_p')$ are given by Equation (8).

**Proof.** Since Algorithm 1 creates a new state in the graph for every job $J_i$ time in Algorithm 1, for any execution scenario such that a job $J_i \in \mathcal{J}$ completes at some time $t$ on core $\phi_k$ (under the given scheduler), there exists a path $P = \langle v_1, \ldots, v_p, v_p' \rangle$ in the schedule-abstraction graph such that $J_i$ is the label of the edge from $v_p$ to $v_p'$ and $t \in [EFT_k(v_p'), LFT_k(v_p')]$, where $EFT_k(v_p')$ and $LFT_k(v_p')$ are given by Equation (8).

5.2 Soundness of the Fast-Forward Phase

We prove that fast-forwarding will not affect any of the successor states of an updated state.

**Lemma 4.** Updating the core intervals of vertex $v_p$ during the fast-forwarding phase does not affect any of the states reachable from $v_p$.

**Proof.** Let $v_p$ be the original state and $v_q$ be the updated state after applying (13). Let path $P$ denote the path from $v_1$ to $v_p$. Note that state $v_q$ shares the same path $P$ as $v_p$. We show that for any arbitrary job $J_i \in \mathcal{J} \setminus \mathcal{J}^P$ (i.e., those that are not scheduled in path $P$) and any arbitrary core $\phi_k(v_p) \in v_p$, the EST and LST of job $J_i$ is the same as for core $\phi_k(v_q)$ in $v_q$. From this we conclude that all system states reachable from $v_p$ are reachable from $v_q$ and that those reachable states remain unchanged. More precisely, we show that, $\forall k, (i) \ EST_{i,k}(v_p) = EST_{i,k}(v_q)$ and (ii) $LST_{i,k}(v_p) = LST_{i,k}(v_q)$.

**Claim (i).** From (2), we have $EST_{i,k}(v_p) = \max\{r^\text{min}_i, EFT_k(v_p)\}$. If the EFT of $\phi_k(v_q)$ has not been updated by (13), i.e., $EFT_k(v_p) > t_\text{min}$, then we trivially have $EST_{i,k}(v_q) = EST_{i,k}(v_p)$. Otherwise, if $EFT_k(v_p)$ has been updated, it must be true that $EFT_k(v_p) \leq t_\text{min}$ and $EFT_k(v_q) = t_\text{min}$. In this case, $EST_{i,k}(v_p) = \max\{r^\text{min}_i, t_\text{min}\} = \max\{r^\text{min}_i, EFT_k(v_p)\} = EST_{i,k}(v_p)$. Since $EFT_k(v_p) \leq t_\text{min} \leq r^\text{min}_i$ (from the definition of $t_\text{min}$). Thus, in both cases, $EST_{i,k}(v_p) = EST_{i,k}(v_q)$.

**Claim (ii).** From (13) we know that if the LFT of a core $\phi_k(v_p)$ is being updated, $LFT_k(v_p) < t_\text{min}$ and $LFT_k(v_q) = t_\text{min}$. By definition, $t_\text{min} = \min\{r^\text{min}_x \mid J_x \in \mathcal{J} \setminus \mathcal{J}^P\} \leq \min\{r^\text{max}_x \mid J_x \in \mathcal{J} \setminus \mathcal{J}^P\} = t_{\text{job}}(v_p)$ (the last equality is due to (6)). Moreover, by (5) we have $t_{\text{core}}(v_p) \leq LFT_k(v_p) < LFT_k(v_q) = t_\text{min} \leq t_{\text{job}}(v_p)$ and $t_{\text{core}}(v_q) \leq LFT_k(v_q) = t_\text{min} \leq t_{\text{job}}(v_q)$ (because $t_{\text{job}}$ only depends on path $P$ and $v_p$ and $v_q$ share the same path). Therefore, by (7), $LST_k(v_p) = \min\{t_{\text{high}}(v_p) - 1, \max\{t_{\text{job}}(v_p), t_{\text{core}}(v_p)\}\} = \min\{t_{\text{high}}(v_q) - 1, \max\{t_{\text{job}}(v_q), t_{\text{core}}(v_q)\}\} = \min\{t_{\text{high}}(v_q) - 1, t_{\text{job}}(v_q)\}$. Since $t_{\text{job}}$ and $t_{\text{high}}$ only depend on path $P$, and $v_p$ and $v_q$ share the same path, the LST in both states is identical, i.e., $LST_{i,k}(v_p) = LST_{i,k}(v_q)$.

5.3 Soundness of the Merge Phase

We now establish that merging two states is safe, i.e., it neither removes a possible job sequence from the graph (Corollary 2), nor does it increase the upper bound on the WCRT
Lemma 5. For a vertex \( v'_p \) created by mutating \( v_p \), any job \( J_i \) that can be scheduled on core \( \phi_k(v_p) \) according to (1), can still be scheduled on core \( \phi_k(v'_p) \) according to (1).

Lemma 6. Let \( v'_p \) be created by mutating \( v_p \), and let \( v_q \) and \( v'_q \) be the vertices resulting from scheduling job \( J_i \) on core \( \phi_k(v_p) \) and \( \phi_k(v'_p) \), respectively. \( \forall x \), \( LFT_x(v'_q) \leq LFT_x(v_q) \) or \( LFT_x(v_p) \leq t_{job}(v_p) \).

Lemma 7. Let \( v'_p \) be created by mutating \( v_p \), and let \( v_q \) and \( v'_q \) be the vertices resulting from scheduling job \( J_i \) on core \( \phi_k(v_p) \) and \( \phi_k(v'_p) \), respectively. \( \forall x \), \( EFT_x(v'_q) \leq EFT_x(v_q) \).

Lemma 8. If \( v'_p \) is a vertex created by mutating \( v_p \), then all the system states reachable from \( v_p \) are also reachable from \( v'_p \).

Lemma 9. Let \( v_q \) and \( v_p \) be two vertices such that \( J^P = J^Q \) (i.e., the set of jobs scheduled until reaching \( v_q \) is equal to the set of jobs scheduled until reaching \( v_p \)), then the state \( v'_p \) resulting from merging \( v_p \) and \( v_q \) with Algorithm 2 is a mutated version of both \( v_p \) and \( v_q \).

By successively applying Lemmas 8 and 9, we obtain the following corollary.

Corollary 1. Let \( v_q \) and \( v_p \) be two vertices such that \( J^P = J^Q \) (i.e., the set of jobs scheduled until reaching \( v_q \) is equal to the set of jobs scheduled until reaching \( v_p \)), all system states reachable from \( v_p \) and \( v_q \) are also reachable from the merged state \( v'_p \).

Corollary 2. For two states that are merged by Algorithm 1, all system states reachable from either of them are also reachable from the merged state.

Proof. Since for two states \( v_p \) and \( v_q \), Definition 2 enforces that \( J^P = J^Q \), the resulting merged state satisfies the requirement of Corollary 1 and hence proves the claim.

5.4 Soundness of Algorithm 1

By successively applying Lemmas 3 and 4 and then Corollary 2, we obtain that the analysis is safe, as stated in Theorem 1 and its corollary below.

Theorem 1. For any execution scenario such that a job \( J_i \in \mathcal{J} \) completes at some time \( t \) on core \( \phi_k \) (under the given scheduler), there exists a path \( P = (\langle v_1, \ldots, v_p, v'_p \rangle) \) in the schedule-abstraction graph such that \( J_i \) is the label of the edge from \( v_p \) to \( v'_p \), and \( t \in [EFT_k(v'_p), LFT_k(v'_p)] \), where \( EFT_k(v'_p) \) and \( LFT_k(v'_p) \) are given by Equation (8).

Corollary 3. Lines 10 and 11 of Algorithm 1 calculate a lower and an upper bound on the BCRT and WCRT, respectively, of every job in \( \mathcal{J} \).
**Proof.** Lines 10 and 11 obtain a job’s response time directly from (8), which provides correct bounds on the earliest and latest finish times of a job according to Lemma 1. Since according to Theorem 1, for any execution scenario, there is a path in the graph, Algorithm 1 includes all possible schedules of a job and hence the obtained values are correctly lower-bounding and upper-bounding the actual BCRT and WCRT of that job.

5.5 Inexactness of Algorithm 1

The following example shows that the abstraction that we use to represent core states may reflect impossible execution scenarios. Therefore, Algorithm 1 is sufficient but not exact.

Assume that a system state $v_p$ contains two core intervals $\phi_1 = [5, 10]$ and $\phi_2 = [1, 10]$ and that there is an unscheduled job $J_1$ with $C_{\min}^1 = C_{\max}^1 = 5$, $r_{\min}^1 = r_{\max}^1 = 1$, and $d_1 = 30$. Further, assume that during the expansion phase of Algorithm 1, $J_1$ is dispatched to $\phi_1$, which results in $\phi_1 = [10, 15]$ and $\phi_2 = [5, 10]$ (after the update phase). According to this new system state, it may happen that core $\phi_2$ becomes available at time $5 \in [5, 10]$, and that core $\phi_1$ remains busy until time $15 \in [10, 15]$. However, this scenario is actually impossible. If $\phi_1$ remains busy until time 15, then $J_1$ must have started to execute at time 10, implying that both $\phi_1$ and $\phi_2$ must have been busy until time 10. Otherwise, job $J_1$ would have been dispatched on $\phi_2$ rather than $\phi_1$. In other words, $\phi_1$ may become available at time 15 only if $\phi_2$ becomes available no earlier than time 10. This example shows a dependency between the availability time of the cores, which is ignored in the current system state abstraction to keep the system state encoding simple, and to increase the number of states that can be merged. This design decision, however, makes the analysis inexact since it considers all possible but also some impossible execution scenarios.

6 Empirical Evaluation

We conducted experiments to answer two main questions: (i) does our test yield better schedulability; and (ii) is the runtime of our analysis practical? To answer the first question, we applied Algorithm 1 to two global non-preemptive scheduling policies: G-NP-FP and G-NP-EDF. As we are unaware of any schedulability analysis for non-preemptive job sets (or periodic tasks) for the aforementioned global scheduling policies, we used the existing tests designed for sporadic non-preemptive task sets as a baseline. These tests include the schedulability test of Baruah [4] for G-NP-EDF (denoted by Baruah-EDF), two tests of Guan et al. [10] for any global non-preemptive work-conserving scheduler (denoted by Guan-Test1-WC), and for G-NP-FP (denoted by Guan-Test2-FP), and the recent schedulability test of Lee (denoted by Lee-FP) [13]. For the sake of comparison, we used simple rate-monotonic priorities for the fixed-priority tests since we did not observe substantial differences when trying out other heuristics such as laxity-monotonic priorities.

To randomly generate a periodic task set with $n$ tasks and a given utilization $U$, we first randomly generated $n$ period values in the range $[10000, 100000]$ microseconds with log-uniform distribution (and a granularity of 5000µs as suggested by Emberson et al. [8]. We then used the RandFixSum [22] algorithm to generate $n$ random task-utilization values that sum to $U$. From the task utilization, we obtained $C_{\max}^i$ and set $C_{\min}^i$ to be $0.1 \cdot C_{\max}^i$. Tasks were assumed to have implicit deadlines. We discarded any task set that had more than 100000 jobs per hyperperiod. Although, in theory, a hyperperiod may contain many more jobs, in industrial settings, e.g., automotive systems [12], periods are usually chosen such that the hyperperiod includes only at most a couple of thousand jobs.
This roughly represents jitter magnitudes that can be expected due to interrupt handling 
the v2 processor clocked at 3 GHz and 1.5 TiB RAM. In the experiments, a task set was claimed 
Schedulability results. Figs. 3-(a) to (c) show a significant gap between the schedulability 
(i) the total system utilization $U$ (for 4 cores 
(ii) the number of tasks $n$ (for 4 cores and $U = 2.8$, which is 70% of the 
(iii) the number of cores $m$ (for 10 tasks and $U = 2.8$), and 
(iv) the total task utilization $U$ while tasks had 100 microseconds release jitter (10 tasks and 4 cores). This roughly represents jitter magnitudes that can be expected due to interrupt handling delays. For each combination of $n$, $m$, and $U$, 1000 random task sets were generated.

To evaluate schedulability of a task set, we implemented Algorithm 1 as a single-threaded C++ program and performed the analysis on a cluster of hosts having an Intel Xeon E7-8857 v2 processor clocked at 3 GHz and 1.5 TiB RAM. In the experiments, a task set was claimed unschedulable as soon as either an execution scenario with a deadline miss was found or a timeout of four hours was reached. Fig. 3 reports the observed schedulability ratio and runtime of Algorithm 1 for different setups. The schedulability ratio is the ratio of task sets deemed to be schedulable divided by the number of generated task sets.

**Schedulability results.** Figs. 3-(a) to (c) show a significant gap between the schedulability ratio of our solution and the state-of-the-art tests. For example, while Lee-FP could only...
identify 8% of schedulable task sets for $U = 2.4$, our test shows that at least 72% of them are schedulable. Similar patterns are seen when the number of tasks or cores increases. As shown in Fig. 3-(b), schedulability improves as the number of tasks increases. This is because, by keeping utilization constant, an increase in $n$ will decrease per-task utilization, which in turn reduces the WCETs and blocking. Thus, more task sets become schedulable. Of the existing tests, however, only the Lee-FP and Guan-Test2-FP tests could benefit from this behavior and only by up to 16% (for $n = 30$).

With the increase in the number of cores, blocking scenarios caused by tasks with large execution times are less likely to occur and hence more task sets are deemed schedulable. However, as shown in Fig. 3-(c), the current tests are quite pessimistic, e.g., Lee-FP could identify only 11% of the task sets as schedulable when (at least) 82% of the task sets are schedulable on 5 cores. From Figs. 3-(a) to (c), we conclude that our analysis is able to reclaim a large portion of pessimism in the baseline analyses (when applied to periodic tasks).

Fig. 3-(d) shows the effect of jitter on schedulability. Since jitter increases the number of possible interleavings between the start time of the tasks, more blocking scenarios become possible and hence tasks with tight deadlines may become unschedulable. This behavior can be observed in the average runtime of the analysis reported in Fig. 3-(h). Yet, our analysis achieves a substantially higher schedulability ratio than the baselines.

It is worth noting that for $U = 0.4$, the counterintuitive drop in schedulability for tasks with jitter is due to the timeout. The bar chart shown at the bottom of Fig. 3-(d) represents the ratio of task sets that could not be analyzed within the four-hour limit. The reason is that for $U = 0.4$, tasks have a small WCET and thus more combinations of job orderings may require analysis before Algorithm 1 is able to merge the branches. In the future, we plan to develop techniques to handle lower (or higher) utilization tasks differently, e.g., by designing more eager merge rules that combine paths with different job sets.

Moreover, we observed that the gap between the schedulability ratio of EDF and FP is small because most of the deadline misses are due to the work-conserving nature of the policy rather than the priority assignment. Namely, since a work-conserving scheduler cannot leave the processor idle, it will schedule any lower-priority job before the next higher-priority job is released. As a result, high-frequency tasks with tight deadlines will miss their deadline before the priority assignment method can play a significant role in improving the order of executions. We conclude that there is a need for a global scheduling algorithm that is able to avoid such blocking scenarios, for instance by being non-work-conserving. While such non-work-conserving non-preemptive scheduling algorithms have recently been proposed for uniprocessor systems [17, 18], currently no such solution exists for multiprocessor platforms.

**Runtime of the analysis.** As shown in Fig. 3-(e), with the increase in the utilization the average runtime of the analysis increases since busy windows become longer. Consequently, paths that have the same set of jobs are merged only at later stages. For larger utilizations such as for $U \geq 2.8$, however, identifying unschedulable task sets becomes easy due to the presence of tasks with large WCETs that can block all cores for a long time. Since we stop the analysis as soon as a deadline miss is found, not-schedulable task sets with large utilization can be identified quickly. The runtime of the analysis hence decreases rapidly for larger utilization values.

Figs. 3-(f) and (g) show that the runtime of the analysis grows with increases in the number of tasks or cores because more states will be generated in the expansion phase. It is worth noting that unlike the effect pertaining to the number of tasks, increasing the number of cores will not increase the runtime monotonically. The reason is that, as shown in Fig. 3-(c), for a workload with $U = 2.8$ and 10 tasks, almost all task sets are schedulable.
on 6 cores or more. That is, the number of cores per se only has a limited effect on the runtime of the algorithm; however, larger platforms are likely to host large task sets, with a potentially large number of jobs per hyperperiod, and our analysis is sensitive to such increases in workload size.

Figs. 3-(i) and (j) report the runtime of the analysis for each task set w.r.t. the number of jobs in a hyperperiod for two scenarios: varying utilization and varying the number of tasks, respectively. As shown by the figures, the runtime of the analysis grows with the increase in the number of jobs in a hyperperiod. We also observe that with an increase in the number of tasks from 10 (Fig. 3-(i)) to up to 30 (Fig. 3-(j)), the largest observed runtime of the analysis grows linearly, i.e., from 1000 to 4000.

Since a naive analysis without path merging does not scale even for a uniprocessor system, as shown in [16], we did not perform a separate experiment to show the efficiency of the path merging technique. In the future, we plan to further explore the design space for different merge conditions and their efficiency for different task set types and utilizations.

Benefits. Overall, we conclude that: (i) the proposed analysis is practical for realistic workload sizes, (ii) it identifies a significantly larger portion of schedulable tasks in comparison with state-of-the-art tests for sporadic tasks, and (iii) even when jitter is considered (which allows for more blocking scenarios and uncertainties), our analysis still achieves much higher schedulability than the baseline tests.

Limitations. We also observed that the runtime of the analysis grows quickly (e.g., more task sets hit the four-hour timeout) for larger systems, e.g., when more than 20 tasks run on a 16 core platform. This is due to the increase in the number of tasks and the number of ways a task can be assigned to a core in the expansion phase of the algorithm. To scale to such large systems, a more efficient abstraction will be needed that allows for more eager merging techniques.

7 Conclusion

The paper provides a sufficient schedulability analysis for global job-level fixed-priority scheduling algorithms and non-preemptive job sets. We have presented a technique for deriving an upper bound on the WCRT and a lower bound on the BCRT by exploring an abstraction of all possible schedules of a job set that reflects the uncertainties in job execution and release times. We developed the notion of a schedule-abstraction graph for global schedulers and introduced two key techniques, namely path merging and fast-forwarding, to slow the state-space growth and proved the analysis to be sound.

Empirical evaluations on periodic task sets show a significant improvement in identifying schedulable task sets w.r.t. the state-of-the-art tests in all experimental setups. The evaluations show that the runtime of the analysis ranges from a couple of seconds to a couple of hours for realistic system setups, e.g., up to 30 tasks, up to 9 cores, and up to 100000 jobs per hyperperiod, which is an acceptable performance for an offline, design-time analysis.

The experiments reported in the paper are all based on a sequential implementation. We expect that the results could still be improved by parallelizing the analysis so that naturally independent scenarios are explored in parallel. To this end, we hope to derive rules that allow maximum parallelism between independent exploration frontiers. Moreover, we will investigate different merge rules to reduce the runtime of the analysis. We also plan to extend the solution presented here to analyze systems with more complicated properties such as precedence constraints and preemption points, and to other scheduling problems such as gang scheduling.
Appendix

Lemma 5. For a vertex $v_p'$ created by mutating $v_p$, any job $J_i$ that can be scheduled on core $\phi_k(v_p)$ according to (1), can still be scheduled on core $\phi_k(v_p')$ according to (1).

Proof. We must show that (1) holds for $J_i$ and $\phi_k(v_p')$, i.e., $EST_{i,k}(v_p') \leq LST_i(v_p')$. For a mutated state, we have $EFT_i(v_p') \leq EFT_i(v_p)$ which leads to having $EST_{i,k}(v_p') \leq EST_{i,k}(v_p)$ (from (2) since the release time $r^*_{\min}$ of $J_i$ does not change). On the other hand, $LST_i(v_p') = \min\{t_{\text{high}}(v_p') - 1, \max\{t_{\text{job}}(v_p'), t_{\text{core}}(v_p')\}\}$. Since mutating a state will not change the set of scheduled jobs in the path reaching to that state, $t_{\text{job}}(v_p') = t_{\text{job}}(v_p)$ and $t_{\text{high}}(v_p') = t_{\text{high}}(v_p)$. Regarding $t_{\text{core}}(v_p)$ and $t_{\text{core}}(v_p')$, two cases must be considered; by definition of an mutated vertex, either $\exists x$, $LFT_x(v_p') \leq LFT_x(v_p')$, or $\exists \phi_x$ such that $LFT_x(v_p') < LFT_x(v_p) \leq t_{\text{job}}(v_p)$.

Case (i). If there is a core interval $\phi_x$ such that $LFT_x(v_p') < LFT_x(v_p) \leq t_{\text{job}}(v_p)$, then by (5), $t_{\text{core}}(v_p') \leq t_{\text{job}}(v_p)$ and $t_{\text{core}}(v_p') \leq t_{\text{core}}(v_p)$ (recall that $t_{\text{job}}(v_p) = t_{\text{job}}(v_p')$). Therefore, we have $LST_i(v_p') = \min\{t_{\text{high}}(v_p') - 1, t_{\text{job}}(v_p')\}$ and $LST_i(v_p') = \min\{t_{\text{high}}(v_p') - 1, t_{\text{job}}(v_p')\}$, which implies $LST_i(v_p) = LST_i(v_p')$.

Case (ii). If $\forall x$, $LFT_x(v_p) \leq LFT_x(v_p')$, then by (5), $t_{\text{core}}(v_p') \leq t_{\text{core}}(v_p)$, implying by (7) that $LST_i(v_p')$ can only be larger than $LST_i(v_p)$.

Finally, because (1) holds for $v_p$, we have $EST_{i,k}(v_p') \leq EST_{i,k}(v_p) \leq LST_i(v_p') \leq LST_i(v_p)$. Thus, Condition (1) holds for $J_i$ and core $\phi_k(v_p')$.

Lemma 6. Let $v_p'$ be created by mutating $v_p$, and let $v_q$ and $v_q'$ be the vertices resulting from scheduling job $J_i$ on core $\phi_k(v_p)$ and $\phi_k(v_p')$, respectively. Then we have $\forall x$, $LFT_x(v_q') \geq LFT_x(v_q)$ or $LFT_x(v_q) \leq t_{\text{job}}(v_q)$.

Proof. We know from Lemma 5 that $v_q'$ exists if $v_q$ does. We show that $\forall x$, $LFT_x(v_q') \geq LFT_x(v_q)$ or $LFT_x(v_q) \leq t_{\text{job}}(v_q)$.

Case (i). First consider core $\phi_k$ (i.e., the core on which $J_i$ is scheduled). According to (8), $LFT_k(v_q) = LST_i(v_q) + C_{\text{max}}$ and $LFT_k(v_q') = LST_i(v_q') + C_{\text{max}}$. Since $v_q'$ is a mutated state of $v_q$, similar to cases (i) and (ii) of Lemma 5’s proof, $LST_i(v_q') \geq LST_i(v_q)$. It follows that $LFT_k(v_q') \geq LFT_k(v_q)$. This proves the claim for $\phi_k$.

Case (ii). For all the other cores $\phi_x$ s.th. $x \neq k$, $LFT_x(v_q)$ and $LFT_x(v_q')$ are computed with (9) (during the expansion phase) and (13) (during the fast-forward phase). Therefore, $LFT_x(v_q) = \max\{t_{\text{min}}(v_q), LFT_x(v_q), EST_{i,k}(v_p)\}$ and $LFT_x(v_q') = \max\{t_{\text{min}}(v_q'), LFT_x(v_q'), EST_{i,k}(v_p')\}$. We consider three sub-cases for when $LFT_x(v_q)$ is equal to $t_{\text{min}}(v_q)$, $LFT_x(v_q')$, or $EST_{i,k}(v_p)$:

(ii.a) Assume that $\max\{t_{\text{min}}(v_q), LFT_x(v_q), EST_{i,k}(v_p)\} = t_{\text{min}}(v_q)$. We note that because $t_{\text{min}}(v_q)$ only depends on the jobs that have been scheduled on the path to reaching $v_q$, we have that $t_{\text{min}}(v_q) = t_{\text{min}}(v_q')$ ($v_q$ and $v_q'$ share the same path). Hence $LFT_x(v_q) = t_{\text{min}}(v_q) = t_{\text{min}}(v_q') \leq \max\{t_{\text{min}}(v_q'), LFT_x(v_q'), EST_{i,k}(v_p')\} = LFT_x(v_q')$. This proves the claim for this case.

(ii.b) If $\max\{t_{\text{min}}(v_q), LFT_x(v_q), EST_{i,k}(v_p)\} = EST_{i,k}(v_p)$, then $EST_{i,k}(v_p) \geq LFT_x(v_q)$ (the last inequality comes from (5)). Further, by Condition (1), job $J_i$ was eligible to be scheduled on $\phi_k(v_p)$ only if $EST_{i,k}(v_p) \leq \min\{t_{\text{high}}(v_p) - 1, \max\{t_{\text{job}}(v_p), t_{\text{core}}(v_p)\}\}$. Because $EST_{i,k}(v_p) \geq t_{\text{core}}(v_p)$, it must hold that $EST_{i,k}(v_p) \leq t_{\text{job}}(v_p)$. Since by assumption $LFT_x(v_q) = EST_{i,k}(v_p)$, we have $LFT_x(v_q) \leq t_{\text{job}}(v_p) \leq C_{\text{max}} \leq \min\{r_{\text{max}}^J \mid J \in J \setminus J^P\}$, which proves the claim for this case.
Finally, if \( \max \{t_{\min}(v_q), \text{LFT}_x(v_p), \text{EST}_{i,k}(v_p)\} = \text{LFT}_x(v_p) \), then, because \( v_q' \) is a mutated version of \( v_p \), we either have \( \text{LFT}_x(v_q') \geq \text{LFT}_x(v_p) \) or \( \text{LFT}_x(v_p) \leq t_{\text{job}}(v_p) \). If \( \text{LFT}_x(v_p') \geq \text{LFT}_x(v_p) \), then \( \text{LFT}_x(v_q') = \max \{t_{\min}(v_q'), \text{LFT}_x(v_p'), \text{EST}_{i,k}(v_p')\} \geq \text{LFT}_x(v_p) \). If \( \text{LFT}_x(v_p') \leq t_{\text{job}}(v_p) \), then \( \text{LFT}_x(v_q') = \text{LFT}_x(v_q) \), which proves the claim. If \( \text{LFT}_x(v_p') \leq t_{\text{job}}(v_p) \), then \( \text{LFT}_x(v_q') = \text{LFT}_x(v_p') \leq t_{\text{job}}(v_p) \leq \min \{r^\text{max}_q \mid J_y \in J \setminus \mathcal{J}^P \} \leq \min \{r^\text{max}_q \mid J_y \in J \setminus \mathcal{J}^P \} \). ▶

**Lemma 7.** Let \( v_q' \) be created by mutating \( v_p \), and let \( v_q \) and \( v_q' \) be the vertices resulting from scheduling job \( J_i \) on core \( \phi_k(v_p) \) and \( \phi_k(v_p') \), respectively. Then we have \( v_x \in \text{EFT}_x(v_q') \leq \text{EFT}_x(v_q) \).

**Proof.** We know from Lemma 5 that \( v_q' \) exists if \( v_q \) does. We show that \( v_x \in \text{EFT}_x(v_q') \leq \text{EFT}_x(v_q) \).

**Case (i).** First consider core \( \phi_k(v_q') \) (i.e., the core on which \( J_i \) is scheduled). According to (8) and (2), \( \text{EFT}_x(v_q) = \text{EST}_{i,k}(v_q) + C_i^{\min} = \max \{r_{\min}^{i}, \text{EFT}_x(v_p)\} + C_i^{\min} \) and \( \text{EFT}_x(v_q') = \text{EST}_{i,k}(v_q') + C_i^{\min} = \max \{r_{\min}^{i}, \text{EFT}_x(v_p')\} + C_i^{\min} \). Since \( v_q' \) is a mutated version of \( v_q \), we have that \( \text{EFT}_x(v_q') \leq \text{EFT}_x(v_q) \), and, inserting it in the above equations, \( \text{EFT}_x(v_q') \leq \text{EFT}_x(v_q) \).

**Case (ii).** For the other core intervals \( \phi_x(v_q') \) s.t. \( x \neq k \), \( \text{EFT}_x(v_q) \) and \( \text{EFT}_x(v_q') \) are computed with (9) (during the expansion phase) and (13) (during the fast-forward phase). Therefore, \( \text{EFT}_x(v_q) = \max \{t_{\min}(v_q), \text{EFT}_x(v_p), \text{EST}_{i,k}(v_p)\} \) and \( \text{EFT}_x(v_q') = \max \{t_{\min}(v_q'), \text{EFT}_x(v_p'), \text{EST}_{i,k}(v_p')\} \). Because \( v_q' \) is a mutated version of \( v_p \), we have \( \text{EFT}_x(v_q') \leq \text{EFT}_x(v_p') \) which also leads to having \( \text{EST}_{i,k}(v_q') \leq \text{EST}_{i,k}(v_p') \) (from (2) since the release time \( r_{\min}^{i} \) of \( J_i \) does not change). Finally, because \( t_{\min}(v_q) \) only depends on the jobs that have been scheduled on the path to reaching \( v_q \), we have that \( t_{\min}(v_q) = t_{\min}(v_q') \), and \( v_q' \) share the same path. Putting it all together we get \( \text{EFT}_x(v_q') = \max \{t_{\min}(v_q'), \text{EFT}_x(v_p'), \text{EST}_{i,k}(v_p')\} \) and \( \text{EFT}_x(v_q) \), \( \text{EST}_{i,k}(v_p) \) is \( \text{LFT}_x(v_q) \).

**Lemma 8.** If \( v_q \) is a vertex created by mutating \( v_p \), then all the system states reachable from \( v_q \) are also reachable from \( v_p' \).

**Proof.** By Lemma 5, all direct successor states \( v_q \) of \( v_p \) obtained by scheduling a job \( J_i \) on core \( \phi_k(v_p) \) are also reachable from \( v_p' \). Let \( v_q' \) be the successor of \( v_p' \) obtained by scheduling a job \( J_i \) on core \( \phi_k(v_p') \), by Lemmas 6 and 7, state \( v_q' \) is either equal to \( v_q \) or is a mutated version of \( v_q \). Therefore, by inductively applying Lemmas 5, 6 and 7 on \( v_p \) and \( v_p' \) and all their direct and transitive successors, the claim follows. ▶

**Lemma 9.** Let \( v_q \) and \( v_p \) be two vertices such that \( \mathcal{J}^P = \mathcal{J}^Q \) (i.e., the set of jobs scheduled until reaching \( v_q \) is equal to the set of jobs scheduled until reaching \( v_p \)), then the state \( v_p' \) resulting from merging \( v_p \) and \( v_q \) with Algorithm 2, is a mutated version of both \( v_p \) and \( v_q \).

**Proof.** From line 3 of Algorithm 2, we have \( \forall x, \text{EFT}_x(v_p') = \min \{\text{EFT}_x(v_p), \text{EFT}_x(v_q)\} \) and \( \text{LFT}_x(v_p') = \max \{\text{LFT}_x(v_p), \text{LFT}_x(v_q)\} \). Therefore, \( \forall x, \text{EFT}_x(v_p') = \text{EFT}_x(v_p) \) and \( \text{EFT}_x(v_p') \leq \text{EFT}_x(v_q) \) and \( \text{LFT}_x(v_p') \geq \text{LFT}_x(v_q) \) and \( \text{LFT}_x(v_p') \geq \text{LFT}_x(v_q) \). Further, by assumption, \( \mathcal{J}^P = \mathcal{J}^Q \), hence it follows that \( v_p' \) is a mutated version of both \( v_p \) and \( v_q \). ▶
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