Sporadic Multiprocessor Scheduling with Few Preemptions

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Abstract

Consider the problem of scheduling a set of n sporadically arriving tasks with the goal of meeting deadlines on a computer system comprising m processors. Processors are identical. A task $\tau_i$ is characterized by its minimum inter-arrival time $T_i$ and its execution time $C_i$. Tasks can be preempted and they can migrate between processors. We propose an algorithm with utilization bound no lower than 88% and it generates few preemptions.
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Abstract

Consider the problem of scheduling a set of $n$ sporadically arriving tasks with the goal of meeting deadlines on a computer system comprising $m$ processors. Processors are identical. A task $\tau_i$ is characterized by its minimum inter-arrival time $T_i$ and its execution time $C_i$. Tasks can be preempted and they can migrate between processors. We propose an algorithm with utilization bound no lower than 88% and it generates few preemptions.

1. Introduction

Consider the problem of preemptive scheduling of $n$ sporadically arriving tasks on a computer system comprising $m$ processors. A task $\tau_i$ is given a unique index in the range 1...$n$ and a processor is given a unique index in the range 1...$m$. A task $\tau_i$ generates a (potentially infinite) sequence of jobs. The time when these jobs arrive cannot be controlled by the scheduling algorithm and the time of a job arrival is unknown to the scheduling algorithm before the job arrives. It is assumed that the time between two consecutive jobs from the same task $\tau_i$ is at least $T_i$. Every job released from task $\tau_i$ requests to finish the execution of $C_i$ time units at most $T_i$ time units after its arrival. It is assumed that $0 \leq C_i \leq T_i$ and $T_i$ and $C_i$ are real numbers. A processor can execute at most one job at a time, and a job cannot execute on two or more processors simultaneously. The utilization is defined as $U = \frac{1}{m} \cdot \sum_{i=1}^{n} \frac{C_i}{T_i}$. The utilization bound $UB_A$ of an algorithm $A$ is the maximum number such that if $U \leq UB_A$ then all tasks meet their deadlines when scheduled by $A$.

Multiprocessor scheduling algorithms are often categorized as partitioned or global scheduling [11, 9, 13]. Global scheduling algorithms store tasks that have arrived but not finished their execution in one queue which is shared among all processors. At every moment the $m$ highest-priority tasks among the tasks that have arrived but not finished their execution are selected for execution on the $m$ processors. In contrast, partitioned scheduling algorithms partition the set of tasks such that all tasks in a partition are assigned to the same processor. Tasks are not allowed to migrate from one processor to another processor, and hence the multiprocessor scheduling problem is transformed to many uniprocessor scheduling problems. This simplifies scheduling and schedulability analysis because the wealth of results in uniprocessor scheduling can be reused. Unfortunately, all partitioned multiprocessor scheduling algorithms have a utilization bound of 50% or less. Global scheduling can achieve a utilization bound of 100% using a family of algorithms called pfair scheduling [5, 1]. But this great utilization bound comes at a price; all task parameters must be multiples of a time quantum and in every time quantum, a new task is selected for execution. As a result, the number of preemptions can be high [8]. We believe (as do Baker, see [4, page 12]) it is desirable to achieve a high utilization bound without suffering from a large number of preemptions.

In this paper we propose an algorithm for scheduling sporadic tasks. Its utilization bound is no lower than 88%. The algorithm assigns tasks to processors. A few tasks are assigned to two processors and they are carefully dispatched to ensure that they never execute on the two processors simultaneously; the other tasks are assigned to only one processor. This design circumvents the limitation of partitioned scheduling (the utilization bound of 50%) yet, it retains the advantages of partitioned scheduling in that (i) the time complexity of dispatching is independent of the number of processors and (ii) most tasks do not need to migrate at all and if a task is of the type that needs to migrate then it only needs to execute on two processors.

The algorithm generates few preemptions. Let $TMIN$ denote the minimum of all $T_i$. Let $njobs_p(t)$ denote the maximum number of jobs that can arrive during a time interval of length $t$ and are assigned to only execute on processor
p. It holds that during a time interval of length \( t \), the algorithm generates at most \( 12 \cdot \lceil t / T_{MIN} \rceil + 2 + n_{jobs_p}(t) \) preemptions on processor \( p \).

The remainder of this paper is organized as follows. Section 2 discusses the role of preemptions in real-time scheduling. It also discusses how to design a scheduling algorithm with high utilization bound and few preemptions. This discussions leads to our design, which is presented in Section 3 and its performance is proven. Section 4 gives conclusions.

2. Background

We say that a task \( \tau_i \) is preempted at time \( t \) if (i) \( \tau_i \) executed just before time \( t \) (let \( p \) denote this processor) and (ii) \( \tau_i \) did not execute on processor \( p \) just after time \( t \) and (iii) \( \tau_i \) has remaining execution time at time \( t \). With this definition, a job that starts executing is not preempted and a job that finishes executing is not preempted either. Also, observe that a job may execute just before time \( t \) and also just after time \( t \) but these executions are on different processors so with our definition there is a preemption at time \( t \).

We believe this captures the notion of preemption used in the research community.

Preemptions are important to meet deadlines in real-time scheduling on both a uniprocessor and on a multiprocessor. In fact, every non-preemptive scheduling algorithm has a utilization bound of 0% (see Example 2 in [2]).

Although preemption is useful, it is important to not overuse it because a preemption has an associated operating system overhead [15, 14, 12]. The exact cost of a preemption is application and architectural dependent and we will not discuss that (see [6] for an excellent coverage). We will count the number of preemptions and in order to do so we need a metric. One metric could be the maximum number of preemptions that a job can suffer from. This is problematic though because there are task sets for which all scheduling algorithms that meet deadlines cause an infinite number of preemptions per job (one such task set is \( m=1, n=2, T_1=10, C_1=5, T_2=10k, C_2=5k \) and let \( k \) be an integer which approaches infinity). Another metric could be the number of preemptions in a time interval divided by the number of jobs that arrive in the time interval. Unfortunately, this metric is problematic as well because there are task sets sets for which all scheduling algorithms that meets deadlines generate a preemption and no job arrives in this interval; and hence the number of preemptions per job is infinite.

We will express an upper bound on the number of preemptions in a time interval as a function of the duration of this time interval and the number of jobs that arrive in this time interval. For this metric, one can show (see Example 4 in [2]) that an inherent property of pfair scheduling is that the number of preemptions can be large. (The fact that pfair scheduling can cause many preemption has also been pointed out by Devi and Anderson [8].) And that enforcing the pfair constraint only needs to be satisfied when a job arrives was a promising technique to reduce the number of preemptions. The algorithm BF [16] and variants of BF [7, 10] do that. An algorithm known as EKG [3] can be configured to achieve a utilization bound 100% and with even fewer preemptions. This is possible because it requires that only a subset of tasks must satisfy the pfair constraint when jobs arrive. Unfortunately, these algorithms were only designed to schedule periodically arriving tasks; that is, tasks where the time between two consecutive jobs of task \( \tau_i \) is exactly \( T_i \). This model allows algorithms to know, at every time, when the next job arrives and it permits these algorithms to calculate the amount of time that a task should be assigned in the time interval until the next job arrival. But for sporadic tasks, this technique cannot be used: the time of next arrival is unknown. For this reason, we will reason about how to design an algorithm for sporadic tasks; this reasoning takes as starting point partitioned scheduling, and then (in Section 3) we will present the new algorithm based on that reasoning.

It is well-known that partitioned scheduling has a utilization bound of at most 50%. For illustrative purpose, the argument is repeated in Example 1.

**Example 1.** Consider \( n = m + 1 \) tasks \( \tau_i \) with \( T_i = 1 \) and \( C_i = 0.5 + \epsilon \). In partitioned scheduling, tasks cannot migrate; they are assigned to a processor and always execute there. Since \( n > m \), there is one processor which is assigned two or more tasks. Therefore, the utilization of this processor will be at least \( 1 + 2\epsilon \), and this is more than 100%. By choosing \( m \to \infty \) and \( \epsilon \to 0 \) the task set will have \( U_s \to 0.5 \) and a deadline is missed. Hence, the utilization bound of every partitioned scheduling algorithm is 50% or less.

This example stresses the fact that deadlines can be missed simply because a task could not be assigned to a processor, although there was plenty of idle time in the overall system. The idle time was spread out on different processors and could not be used. However, in the same previous example a task is given some utilization on one processor and some utilization on another processor then it is possible to assign tasks such that the utilization on every processor reaches 100%. This approach was used in EKG. For many tasks, it gives utilization to only a single processors but for a few tasks, it gives utilization to two processors. For those tasks, it is imperative however that a task does not execute on two processors simultaneously. This property was ensured for those tasks by giving such tasks execution such that the amount of execution between two consecutive arrivals divided by the time between these two
consecutive arrivals is exactly equal to the utilization of the task on that processor. For sporadically arriving tasks such a scheme is not possible because the next arrival time is unknown. One can however, subdivide the timeline into time slots of duration $S$ and ensure that each task that executes on two processors is given execution proportional to its utilization on these time slots. This approach does not require knowledge of future arrival times.

Figure 1 presents an example that illustrates how to perform such run-time dispatching. The task $r_2$ has $C_2/T_2=0.55$. We can see (in Figure 1(a)) that $r_2$ is assigned a reserve of 40% of the processing capacity of processor $P_1$ and $r_2$ is assigned a reserve of 15% of the processing capacity of processor $P_2$. This processing capacity is distributed in each time interval of duration $S$. If $r_2$ arrives exactly in the beginning of a time slot and if the deadline of $r_2$ is exactly at the end of a time slot then $r_2$ is given enough execution and hence it will meet its deadlines. But for other arrival times this may not be the case. Consider Figure 1(b). Here we can see the arrival time of a job of task $r_2$. For this arrival time, it is necessary that the reserves of $r_2$ are larger in order to meet its deadline. We will let the reserves of a processor assigned to a task be the same for all time slots. Consequently, the reserve on processor 1 given to task $r_2$ will be more than 40% regardless of when and if $r_2$ arrives.

It is necessary to assign a value to $S$ and several choices are possible. Choosing $S$ to be small implies that the reserves only need to be increased by a small amount to meet deadlines and this offers a high utilization bound. On the other hand, choosing $S$ to be large implies that the number of preemptions is small. As a compromise, we choose $S=TMIN/4$.

3. The new algorithm

The new algorithm brings the ideas that were successful for the design of EKG and exploits them for sporadically arriving tasks. The new algorithm consists of two steps. First, tasks are assigned to processors. Some tasks may be assigned to two processors, meaning that this task may execute on any of these two processors but it is not permitted
to execute on both processors simultaneously. We refer to such a task as *split tasks*. Naturally tasks that are not split tasks are referred to as *non-split tasks*. The algorithm for task assignment is presented in Figure 2 and it is performed before run-time. Then, at run-time, the tasks are dispatched; Figure 3 presents this algorithm. Let us now understand the behavior, the rationale of the design of the new algorithm and then prove its performance.

Figure 2 presents the algorithm for assigning tasks to processors. The algorithm assigns tasks to processors such that on all processors the utilization does not exceed 100%. The algorithm treats *heavy* and *light* tasks differently. A task \( \tau_i \) is heavy if \( C_i / T_i > SEP \), otherwise it is light. SEP means separator. The exact value of SEP will be specified later. First, the algorithm assigns heavy tasks to their own dedicated processors (at lines 11-16); no other task will be assigned to these processors. The main idea of the algorithm is that there is a current processor, with index \( p \) and tasks are considered one by one; index \( i \) denotes the current task. The task currently under consideration is attempted to be assigned to the current processor \( p \). This is performed at line 19. If the condition stated in that line is true then the task can be assigned to processor \( p \). If the condition is false, the utilization of the task is assigned to two processors (at lines 25-30) and the task assigned to the current processor \( p \) and processor \( p + 1 \). Then the processor with a higher index is considered (line 31). As already pointed out, the algorithm assigns heavy tasks to some processors and light tasks to some other processors. \( L \) separates these tasks: heavy tasks are assigned to processors with indices ranging from 1 up to \( L \), while light tasks are assigned to processors with indices ranging from \( L + 1 \) up to \( m \).

The performance of the task assignment scheme (presented in Figure 2) is given by Lemma 1.

**Lemma 1.** If \( 0 \leq SEP \leq 1 \) and a task set satisfies \( U_s \leq SEP \) and the algorithm in Figure 2 is used then the algorithm in Figure 2 declares SUCCESS.

**Proof.** The proof is by contradiction. Suppose that the algorithm still declares failure. Let us delete all tasks with a higher index than \( i \). We obtain that the algorithm still declares failure. If there was any task with \( C_j / T_j > SEP \) then this task is deleted and this processor is deleted. If we rerun the algorithm with the new task set then it still declares failure. We can repeat this argument until it holds for all tasks that \( \forall j : C_j / T_j \leq SEP \). Consequently we now have a task set \( \{ \tau_1, \tau_2, \ldots, \tau_n \} \) that satisfies

\[
U_s \leq SEP
\]  

and \( \forall j : C_j / T_j \leq SEP \) and the algorithm in Figure 2 is used and the algorithm in Figure 2 declares FAILURE. From line 19 and lines 25-30 we obtain:

\[
\forall p \in 1 \ldots m - 1 : U[p] = SEP
\]

and

\[
U[m] + C_n / T_n > SEP
\]  

Figure 2. An algorithm for assigning tasks to processors.

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1. for \( p \) in \( 1..m \) do
2. \( U[p] := 0 \)
3. \( \text{lo\_split}[p] := 0 \)
4. \( \text{hi\_split}[p] := 0 \)
5. end for
6. Let \( \tau_{\text{heavy}} \) denote the set of tasks such that \( C_i / T_i > SEP \)
7. Let \( \tau_{\text{light}} \) denote the set of tasks such that \( C_i / T_i \leq SEP \)
8. \( L := \lceil \text{heavy} \rceil \)
9. Order tasks such that \( \tau_i \) with \( i \) in \( 1..L \) are all in \( \tau_{\text{heavy}} \)
10. and \( \tau_i \) with \( i \) in \( L+1..m \) are all in \( \tau_{\text{light}} \)
11. Sort tasks in \( L+1..m \) such that \( T_{L+1} \leq T_{L+2} \leq \ldots \leq T_m \)
12. for \( i \) in \( 1..L \) do
13. \( j := i \)
14. \( U[p] := U[p] + C_i / T_i \)
15. \( \tau_i, \text{processor}\_id1 := p \)
16. end for
17. \( p := L + 1 \)
18. for \( i := L+1 \) to \( m \) do
19. if \( U[p] + C_i / T_i \leq SEP \) then
20. \( U[p] := U[p] + C_i / T_i \)
21. \( \tau_i, \text{processor}\_id1 := p \)
22. \( \tau_i, \text{processor}\_id2 := p \)
23. end if
24. if \( p+1 \leq m \) then
25. \( \text{hi\_split}[p] := SEP - C_i / T_i \)
26. \( \text{lo\_split}[p+1] := C_i / T_i - \text{hi\_split}[p] \)
27. \( \tau_i, \text{processor}\_id1 := p \)
28. \( \tau_i, \text{processor}\_id2 := p+1 \)
29. \( U[p] := U[p] + \text{hi\_split}[p] \)
30. \( U[p+1] := U[p+1] + \text{lo\_split}[p+1] \)
31. \( p := p+1 \)
32. else
33. declare FAILURE
34. end if
35. end for
36. end if
37. declare SUCCESS
Combining Equality 2 and Inequality 3 and using the knowledge that $\tau_1, \tau_2, \ldots, \tau_{n-1}$ are assigned on processors $P_1, P_2, \ldots, P_m$ yields:

$$\sum_{i=1}^{n} C_i / T_i > SEP \cdot m$$  \hfill (4)

Dividing by $m$ and using the definition of $U_s$ yields

$$U_s > SEP$$  \hfill (5)

This contradicts Inequality 1 and hence the lemma is correct.

We are now in position to see a precise statement of the dispatcher used at run-time. Figure 3 presents this algorithm. There is a constant, $\alpha$ which will be specified later in this paper. It is assumed that if the array $\text{split\_tasks}$ is indexed out of bounds then it returns NULL. It is also assumed that if $\text{switch\_to}$ is called with a NULL parameter then the processor is kept idle.

The variable $rq$ (on line 11), means run-queue and it is a variable which store tasks that are ready for execution or are executing. This ready queue consists of two types of tasks (i) tasks that may execute on two different processors (called $\text{split\_tasks}$) and (ii) tasks that may only execute on a single processor (called $\text{non\_split\_tasks}$). It can be seen (from the algorithm that assigns tasks to processors, in Figure 2) that at most $m$ tasks are $\text{split\_tasks}$ and there can be at most one task split between processor $p$ and $p+1$. For this reason, we only need an array of $m$-1 elements for storing the split tasks. For the non-split tasks, however, it is necessary to store a priority queue for each processor (see line 8). This priority queue (line 6) contains data structures of the type $\text{task\_struct}$ and they are ordered in EDF order. On line 24 in Figure 3, a task struct is inserted in a queue; it is assumed that if this task struct is already in the queue then this operation takes no effect. Analogously, on line 34 in Figure 3, a task struct is removed from a queue; it is assumed that if this task struct is not in the queue then this operation takes no effect.

Each task is represented in the operating system by a variable of the type $\text{task\_struct}$ (shown in lines 1-5). The notation $\tau_i$ refers to the task struct of task $\tau_i$. The variables $\text{processor\_id1}$ and $\text{processor\_id2}$ state which processor a task is assigned to. If a task is only assigned to one processor then $\text{processor\_id1}$ and $\text{processor\_id2}$ are set to the id of this processor. The values of $\text{processor\_id1}$ and $\text{processor\_id2}$ are outputs from the algorithm in Figure 2. The variable $T_i$ of a task. The variable $d$ state the absolute deadline of the current job of a task.

The variable $t_0$ and $t_1$ (on line 15) stores the current time slot of length $S$. This is the time slot discussed in Figure 1. $t_0$ is the start time of the time slot, whereas $t_1$ is the finishing time of the time slot. For a processor $p$, the time interval $[t_0, t_1)$ can be divided into three subintervals $[t_0, \text{timea}[p])$, $[\text{timea}[p], \text{timeb}[p])$ and $[\text{timeb}[p], t_1)$. A processor $p$ is said to be in state $a$ if the current time is in $[t_0, \text{timea}[p])$; the processor is in state $x$ if the current time is in $[\text{timea}[p], \text{timeb}[p])$ and the processor is in state $b$ if current time is in $[\text{timeb}[p], t_1)$. There is an array: state. The variable $\text{state}[p]$ stores the state of processor $p$. timea[p] and timeb[p] are computed based on the $\text{lo\_split}$ and $\text{hi\_split}$ (which were computed in the algorithm in Figure 2) and $\alpha$ (which control how much larger the reserve needs to be in order to ensure that split tasks meet their deadlines for all arrival times).

The event handler "when the current time becomes $t_1$ do" calculates the times that govern when processors should perform state transition. In particular, when the time becomes $t_1$ then new values of $t_0, \text{timea}$, $\text{timeb}$ and $t_1$ are calculated (as shown on lines 50-53).

The scheduling is performed by the lines 16-77. It is a state machine executing on each processor and it reacts on events. The main idea is that if a processor $p$ is in state a then it executes the reserve that it shares the processor $p-1$, whereas if a processor $p$ is in state $b$ then it executes the reserve that it shares the processor $p+1$ and if a processor $p$ is in state $x$ then it uses EDF to dispatch a non-split tasks assigned to processor $p$. Observe however that if the task that was intended to be executed on the reserve does not need to execute then the non-split task with the earliest deadline is selected.

We will now assign values to the parameters SEP and $\alpha$. From the task assignment presented in Figure 2, we know that if a task $\tau_i$ is assigned to two processor $p$ and $p+1$ then it holds that:

$$\text{lo\_split}[p + 1] + \text{hi\_split}[p] = \frac{C_t}{T_i}$$  \hfill (6)

For this purpose of understanding how to choose SEP and $\alpha$, Figure 4 is instrumental. Consider task $\tau_i$ in Figure 4(a) and this particular arrival time. From this figure and considering $p=1$ and considering lines 50-53 in Figure 3 we obtain that:

$$x = S \cdot (\alpha + \text{lo\_split}[p + 1])$$  \hfill (7)

and

$$y = S \cdot (\alpha + \text{hi\_split}[p])$$  \hfill (8)

and

$$4 \cdot S + (S - (x + y)) = T_i$$  \hfill (9)

If $\tau_i$ missed a deadline it must have been that it used all the time available in the reserves. That is, it must have been that:

$$4 \cdot (x + y) < C_t$$  \hfill (10)
Combining Inequality 10 and Equality 9 yields:

\[
\frac{4 \cdot (x + y)}{4 \cdot S + (S - (x + y))} < \frac{C_i}{T_i} \tag{11}
\]

Let us apply Equality 7 and Equality 8 on Inequality 11. Then apply Inequality 6 yields:

\[
\frac{4 \cdot (2\alpha + \frac{C_i}{T_i})}{5 - (2\alpha + \frac{C_i}{T_i})} = \frac{C_i}{T_i} < 0 \tag{12}
\]

One can show that if we choose \(\alpha = \frac{9}{2} - 2\sqrt{5} \approx 0.0278\) and let us consider how it impacts a task \(\tau_i\) that is only assigned to a single processor. Figure 4(b) shows a specific scenario. Let us explore how much processor \(p\) in Figure 4(b) can be utilized. From this figure we obtain that:

\[
x = S \cdot (\alpha + lo_{\text{split}}[p]) \tag{13}
\]

and

\[
z = S \cdot (\alpha + hi_{\text{split}}[p]) \tag{14}
\]

\[
4 \cdot S + (x + z) = T_j \tag{15}
\]
(a) The reserves must be large enough to ensure that a task $\tau_i$ that is assigned to two processors receives enough execution even when it arrives at an unfavorable time.

(b) The reserves must not be too large because then they can cause interference on another task $\tau_j$ which is only assigned to a single processor.

Figure 4. How to inflate the reserves properly.

If task $\tau_j$ missed its deadline for the scenario shown in Figure 4(b) then the processor $p$ was busy during the time from when the job $\tau_j$ arrived until its deadline expired. (This is true because $\tau_j$ is a non-split task and non-split tasks are allowed to be executed in state $a$ and $b$ with a lower priority than the tasks that were assigned these reserves. See line 62 and line 68 in the algorithm in Figure 3.) Because of this deadline miss it holds that:

$$5 \cdot (x + z) + C_j > T_j$$  \hspace{1cm} (16)

Dividing Inequality 16 by $T_j$ and applying Equality 15 yields:

$$\frac{5 \cdot (x + z)}{4 \cdot S + (x + z)} + \frac{C_j}{T_j} > 1$$ \hspace{1cm} (17)

Applying Equality 13 and Equality 14 and rewriting yields:

$$\frac{5 \cdot (2\alpha + \text{lo} \_ \text{split}[p] + \text{hi} \_ \text{split}[p])}{4 \cdot (2\alpha + \text{lo} \_ \text{split}[p] + \text{hi} \_ \text{split}[p])} + \frac{C_j}{T_j} > 1$$

One can show that for the $\alpha$ selected and for $\text{lo} \_ \text{split}[p] + \text{hi} \_ \text{split}[p] \geq 0$ it holds that:

$$18 - 8\sqrt{5} + \text{lo} \_ \text{split}[p] + \text{hi} \_ \text{split}[p] + \frac{C_j}{T_j} > 1$$  \hspace{1cm} (18)

Combining these yields:

$$18 - 8\sqrt{5} + \text{lo} \_ \text{split}[p] + \text{hi} \_ \text{split}[p] + \frac{C_j}{T_j} > 1$$ \hspace{1cm} (19)

Rewriting yields:

$$\text{lo} \_ \text{split}[p] + \text{hi} \_ \text{split}[p] + \frac{C_j}{T_j} > 8\sqrt{5} - 17$$ \hspace{1cm} (20)

As a consequence of this reasoning, it follows that if

$$\text{lo} \_ \text{split}[p] + \text{hi} \_ \text{split}[p] + \frac{C_j}{T_j} \leq 8\sqrt{5} - 17$$ \hspace{1cm} (21)

then $\tau_j$ meets its deadline for the scenario depicted in Figure 4(b).

We have now shown that the choice $\alpha = \frac{9}{2} - 2\sqrt{5} \approx 0.0278$ and SEP=$8\sqrt{5}-17 \approx 0.88854382$ causes deadlines to be met in the examples shown in Figure 4(a) and Figure 4(b). We use these parameters in the remainder of the paper and prove the utilization bound of the algorithm. We will do so by first stating lemmas that ensure that (i) heavy tasks meet their deadlines, (ii) split tasks do not execute on two or more processors simultaneously and (iii) split tasks
meet their deadlines. Using these lemmas and reasoning about non-split tasks will give us the utilization bound.

**Lemma 2.** Assume that tasks assigned to processors 1..L are dispatched by any work-conserving algorithm. Then all tasks assigned to processors 1..L meet their deadlines.

**Proof.** Follows from the fact that we assume $0 \leq C_i \leq T_i$ (which was stated in the introduction). 

**Lemma 3.** Assume that tasks are assigned using the algorithm presented in Figure 2 (with SEP=8√5-17) and tasks on processors 1..L are dispatched by any work-conserving uniprocessor scheduling algorithm and tasks on processors L+1..m are dispatched using the algorithm presented in Figure 3 (with $\alpha = \frac{9}{2} - 2\sqrt{5}$ and $S=TMIN/4$). Then it holds for every task $\tau_i$ that $\tau_i$ never executes on two or more processors simultaneously.

**Proof.** If $\tau_i$ is a non-split task then the truth of the lemma is obvious. We will now prove that the lemma is true also for the case when $\tau_i$ is a split task. Suppose that the lemma was incorrect then there must be a time $t$ when a split task $\tau_i$ executed on two processor simultaneously. Observe that the algorithm presented in Figure 2 assigns a task to at most two processors. Hence $\tau_i$ executes on two processors. Let these processors be denoted $p$ and $p+1$. Due to the assumption on the falsity of the lemma it holds that processor $p$ and processor $p+1$ execute $\tau_i$ at time $t$. From the algorithm Figure 3 it follows that $t \leq time[p+1]$ and $time[p] \leq t$. As a consequence, it follows that:

$$time[p] \leq time[p+1] \quad (22)$$

This can obviously be rewritten as:

$$0 \leq time[p+1] - time[p] \quad (23)$$

Using lines 50-53 in the algorithm presented in Figure 3 yields:

$$1 \leq lo_split[p+1] + hi_split[p] + 2 \cdot (\frac{9}{2} - 2\sqrt{5}) \quad (24)$$

From our choice of SEP we have that:

$$lo_split[p+1] + hi_split[p] \leq 8\sqrt{5} - 17 \quad (25)$$

Combining Inequality 25 with Inequality 24

$$1 \leq -8 + 4\sqrt{5} \quad (26)$$

But this is impossible. Hence the lemma is true. 

**Lemma 4.** Assume that tasks are assigned using the algorithm presented in Figure 2 (with SEP=8√5-17) and tasks on processors 1..L are dispatched by any work-conserving uniprocessor scheduling algorithm and tasks on processors L+1..m are dispatched using the algorithm presented in Figure 3 (with $\alpha = \frac{9}{2} - 2\sqrt{5}$ and $S=TMIN/4$). Then it holds that every split task meet its deadline.

**Proof.** Suppose that the lemma was false then there would be a split task that misses a deadline. We can delete all non-split tasks and then we obtain the same schedule because split tasks are executed with higher priority (see the algorithm presented in Figure 3). And hence there is still a split task that misses a deadline. Let us consider the first time that a deadline was missed. Let $\tau_i$ denote the task that released this job. Since $T_i \geq TMIN$ and $S=TMIN/4$ we obtain that: $T_i \geq 4S$. Let $A_i$ denote the time when this job arrived. Consequently the deadline (that was missed) is at time $A_i+T_i$. We know (from the algorithm in Figure 2) that a split task is only permitted to execute on two processors and they have consecutive index. Let $p$ and $p+1$ be denote those processor where $\tau_i$ is permitted to execute. From the algorithm in Figure 2, we also have that:

$$C_i/T_i = hi_split[p] + lo_split[p + 1] \quad (27)$$

Let $x$ and $y$ be denoted as:

$$x = S \cdot (\alpha + lo_split[p + 1]) \quad (28)$$

and

$$y = S \cdot (\alpha + hi_split[p]) \quad (29)$$

Since a deadline was missed for this task $\tau_i$ it implies that (i) $\forall t \in [A_i,A_i+T_i]$ such that at time $t$ processor $p+1$ was in state $A$, it holds that $\tau_i$ executed and (ii) $\forall t \in [A_i,A_i+T_i]$ such that at time $t$ processor $p$ was in state $B$, it holds that $\tau_i$ executed. Consequently, the amount of time that $\tau_i$ is given execution during $[A_i,A_i+T_i]$ is at least:

$$[T_i/S] \cdot (x + y) - \min([T_i/S] \cdot S - T_i, (x + y)) \quad (30)$$

Since less than $C_i$ time units was performed by task $\tau_i$ during $[A_i,A_i+T_i]$ it follows that:

$$[T_i/S] \cdot (x + y) - \min([T_i/S] \cdot S - T_i, (x + y)) < C_i \quad (31)$$

Applying Inequalities 27, 28, 29 and the value of $\alpha$ yields:

$$[T_i/S] \cdot (x + y) - \min([T_i/S] \cdot S - T_i, (x + y)) < C_i \quad (32)$$

We also know from the assumptions of the lemma that:

$$C_i/T_i \leq 8\sqrt{5} - 17 \quad (33)$$

and due to the choice of $S$:

$$4 \cdot S \leq T_i \quad (34)$$
Dividing Inequality 32 by $T_i$ and then subtracting both sides by $C_i/T_i$ yields:

$$\frac{[T_i/S]}{T_i} \cdot (S/T_i) \cdot (C_i/T_i + 9 - 4\sqrt{5}) - \min([T_i/S] \cdot (S/T_i - 1)) \cdot (C_i/T_i + 9 - 4\sqrt{5}) - C_i/T_i < 0 \tag{35}$$

One can show that under the assumption of Inequality 34 it holds that for every $C_i/T_i$ such that $0 \leq C_i/T_i \leq 1$ then

$$0 \leq \frac{[T_i/S]}{T_i} \cdot (S/T_i) \cdot (C_i/T_i + 9 - 4\sqrt{5}) - \min([T_i/S] \cdot (S/T_i - 1)) \cdot (C_i/T_i + 9 - 4\sqrt{5}) - C_i/T_i \tag{36}$$

Inequality 36 contradicts Inequality 35 and hence the lemma is correct. 

We saw in Lemma 4 that all split tasks meet their deadlines. In the following, we will prove that also the non-split tasks meet their deadlines.

**Theorem 1.** Assume that tasks are assigned using the algorithm presented in Figure 2 (with $\text{SEP}=8\sqrt{5}/17$) and tasks on processors $1..L$ are dispatched by any work-conserving uniprocessor scheduling algorithm and tasks on processors $L+1..m$ are dispatched using the algorithm presented in Figure 3 (with $\alpha = \frac{9}{4} - 2\sqrt{5}$ and $S = \text{TMIN} / 4$). If $U_x \leq 8\sqrt{5}/17$ then all deadlines are met and no task executes on two or more processors simultaneously.

**Proof.** From Lemma 2 it follows that all heavy tasks meet their deadlines and clearly they do not execute on two or more processors simultaneously. For the processors $L+1..m$ that schedules light tasks, we know (from the algorithm presented in Figure 2) that $\forall x$: $U[p] \leq 8\sqrt{5}/17$. From Lemma 4 it follows that also all light tasks that are split meet their deadlines and clearly (from Lemma 3) it holds that they do not execute on two or more processors simultaneously.

It remains to prove that on processors $L+1..m$, all non-split tasks meet their deadlines. We will do so now.

Suppose that a non-split task missed its deadline. If there are many jobs that missed a deadline then let us consider the job with the earliest deadline. Let $\tau_i$ denote the task that released that job and let $p$ denote the processor to which $\tau_i$ was assigned. Let $A_i$ denote the time when this job arrived and consequently the deadline of this job is at time $A_i+T_j$.

We know that the processor $p$ is busy during $[A_i,A_i+T_j]$ (This is true because $\tau_i$ is a non-split task and non-split tasks are allowed to be executed in state $a$ and $b$ with a lower priority than the tasks that were assigned these reserves. See line 62 and line 68 in the algorithm in Figure 3.) Let $Q$ denote the latest time before $A_i+T_j$ such that processor $p$ is busy during the time interval $[Q,A_i+T_j]$. We know that this interval $[Q,A_i+T_j]$ is non-empty because $Q \leq A_i$. We can delete all jobs with arrival times before $Q$: the job released by $\tau_i$ is still missed. Let us consider another job released from a non-split task $\tau_j$. If this job arrived later than $A_i+T_j$ then its deadline is later than $A_i+T_j$. Such a job can be deleted and because of EDF scheduling, the job released by $\tau_i$ still misses its deadline. Let $L$ denote the length of the busy period; that is, $L=A_i+T_j-Q$. Observe that $L \geq T_i$ and $T_i \geq 4 \cdot S$ and this gives us:

$$L \geq 4 \cdot S \tag{37}$$

Let $x$ denote $(\text{lo}\_\text{split}[p]+\alpha) \cdot S$ and let $z$ denote $(\text{hi}\_\text{split}[p]+\alpha) \cdot S$. We know that split tasks meet their deadlines and hence an upper bound on the amount of work that is performed by split tasks on processor $p$ during the time interval of length $L$ is at most:

$$\frac{L}{S} \cdot (x + z) + \min(L - \frac{L}{S} \cdot S, x + z) \tag{38}$$

Let $\text{NS}[p]$ denote the set of tasks that are only assigned to processor $p$. Since a deadline was missed and the processor was busy during the time interval of length $L$, it follows that:

$$\left( \sum_{\tau_j \in \text{NS}[p]} \frac{L}{T_j} \cdot C_j \right) + \frac{L}{S} \cdot (x + z) + \min(L - \frac{L}{S} \cdot S, x + z) > L \tag{39}$$

Dividing by $L$ and using the definition of $x$ and $z$ and using the knowledge that $\frac{L}{T_j} \geq \frac{L}{T_\text{MIN}} \cdot \frac{\alpha}{S}$, yields:

$$\left( \sum_{\tau_j \in \text{NS}[p]} \frac{C_j}{T_j} \right) + \frac{S \cdot (\text{lo}\_\text{split}[p] + \text{hi}\_\text{split}[p] + 9 - 4\sqrt{5})}{L} + \min(1 - \frac{L}{S} \cdot \frac{S}{L}, \frac{S \cdot (\text{lo}\_\text{split}[p] + \text{hi}\_\text{split}[p] + 9 - 4\sqrt{5})}{L}) > 1 \tag{40}$$

One can show that for the $\alpha$ selected and for $\text{lo}\_\text{split}[p] + \text{hi}\_\text{split}[p] \geq 0$ and $L \geq 4 \cdot S$ it holds that:

$$18 - 8\sqrt{5} + \text{lo}\_\text{split}[p] + \text{hi}\_\text{split}[p] \geq \frac{L}{S} \cdot \frac{S \cdot (\text{lo}\_\text{split}[p] + \text{hi}\_\text{split}[p] + 9 - 4\sqrt{5})}{L} + \min(1 - \frac{L}{S} \cdot \frac{S}{L}, \frac{S \cdot (\text{lo}\_\text{split}[p] + \text{hi}\_\text{split}[p] + 9 - 4\sqrt{5})}{L}) \tag{41}$$

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Adding these preemptions gives us the theorem.

Proof. In a time interval, there are \( s_{\text{or}} \) jobs and are assigned using the algorithm presented in Figure 2. It holds that during a time interval of length \( t/\alpha \) and tasks to only execute on processor \( p \). It follows that the statement of the theorem is true.

The number of preemptions is stated in Theorem 2.

**Theorem 2.** Assume that tasks are assigned using the algorithm presented in Figure 2 (with \( SE=8\sqrt{5} - 17 \)) and tasks on processors 1...L are dispatched by any work-conserving uniprocessor scheduling algorithm and tasks on processors \( L+1...m \) are dispatched using the algorithm presented in Figure 3 (with \( \alpha = 9/2 - 2\sqrt{5} \) and \( S=TMIN/A \)). Let \( n_{\text{jobs}}(t) \) denote the maximum number of jobs that can arrive during a time interval of length \( t \) and are assigned to only execute on processor \( p \). It holds that during a time interval of length \( t \), the algorithm generates at most \( 12 \cdot [t/TMIN] + 2 + n_{\text{jobs}}(t) \) preemptions on processor \( p \).

**Proof.** In a time interval, there are \( 12 \cdot [t/TMIN] + 2 \) preemptions due to the execution of reserves. There are \( n_{\text{jobs}}(t) \) preemptions due to EDF scheduling in state \( x \). Adding these preemptions gives us the theorem.

4. Conclusions

We have proposed an algorithm for scheduling sporadic tasks. Its utilization bound is 88% and it generates few preemptions. We left open the important question on how to extend this algorithm for sporadic tasks where the deadline of a task is not equal to its minimum inter-arrival time.

References


A. Background: Details

Example 2. The most interesting part of the schedule in this example is shown in Figure A. Consider \( m+1 \) tasks to be scheduled on a multiprocessor with \( m \) identical processors where the scheduling algorithm is non-preemptive. Tasks are permitted to change priorities at any time. Because non-preemptive scheduling is used, the only migration permitted is that jobs of the same task may execute on different processors. We assume that the scheduling algorithm is permitted to insert idle time. Let the tasks be characterized as follows. \( T_1=1 \) and \( C_1=2 \). For all \( i=2..m+1 \): \( T_i=\epsilon^{-i-1}, C_i=2\epsilon^i \). We choose \( \epsilon<1/2 \). We let \( O_i \) denote the time when \( \tau_i \) arrives for the first time. The scheduling algorithm is not permitted to choose \( O_i \). Let us consider an arbitrary job release by \( \tau_i \). Let \( A_i \) denote the arrival time of that job and let \( s_i \) denote the start time of the execution of that job. Because the execution time of \( \tau_i \) is twice as long as \( T_2 \) we know that every time \( \tau_i \) executes, \( \tau_2 \) can execute at least once such that \( s_1 \leq A_2 < A_2 + T_2 \leq s_1 + C_1 \), and hence there is a time when \( \tau_1 \) and \( \tau_2 \) must execute simultaneously. Repeating this argument, we obtain that there is a time when all tasks have to execute. All the \( m+1 \) tasks must execute on different processors and we only have \( m \) processors. Hence a deadline is missed. Here, \( U_s = 2 \cdot (m+1)/m \). Choosing \( \epsilon \to 0 \) yields that the utilization bound is zero. Naturally this example can be used to show that the utilization bound of every non-preemptive uniprocessor scheduling algorithm is zero as well. This reasoning can be performed for both periodically and sporadically arriving tasks. Consequently, the utilization bound of non-preemptive scheduling is zero for any number of processors and for both periodically arriving tasks and sporadically arriving tasks.

Example 3. Consider \( m \) processors and \( n=m+1 \) tasks. The tasks are characterized by \( \forall i \in 1..m+1 \): \( C_i=m/(m+1) \), \( T_i=1 \) and \( m \geq 4 \) and they all arrive simultaneously at time 0. Consider \( 1/(2(m+1)) \), \( 1-1/(2(m+1)) \). In this time interval, a task must execute at least \( m/(m+1)-1/(m+1) \geq 0.6 \) time units in order to meet deadlines. Since each task must execute at least 0.6 time units in this interval, at least one task is preempted in the time interval. Also, no tasks arrive in the time interval. Consequently, the number of preemptions divided by the number of arriving jobs in this interval is infinite.

Example 4. Consider \( n=6 \) tasks to be scheduled on \( n=5 \) processors. We consider the special case when all tasks arrive at time 0 and all tasks arrive periodically. \( \text{primes}(k) \) denotes the \( k \text{th} \) prime number. Let \( T_1=2 \cdot \text{primes}(5)=22, T_2=2 \cdot \text{primes}(6)=26, T_3=2 \cdot \text{primes}(7)=34, T_4=2 \cdot \text{primes}(8)=38, T_5=2 \cdot \text{primes}(9)=46 \) and \( T_6=2 \cdot \text{primes}(10)=54 \). Let \( C_1=13, C_2=15, C_3=19, C_4=21, C_5=24 \) and \( C_6=28 \). This task set has \( U_s=0.6639 \). Let \( \text{lcm} \) denote \( \text{lcm}(T_1,T_2,...,T_n) \). \( \text{lcm} \) is 57366738 and the number of jobs during \([0, \text{lcm})\) is 10320350. We will now compute the number of preemptions. We will use pfair algorithms to schedule this task set and we will assume that the tasks arrive as frequently as permitted by the sporadic model.

The schedule generated by the pfair algorithms, \( PD^2 [5] \), early-release ER pfair [1] and bounded fairness (BF) [16] are illustrated in Figure A. \( PD^2 \) causes 15.47 preemptions/job during \([0, \text{lcm})\). The result is 3.75 for early-release pfair, and 3.82 for bounded fairness. We have selected this example for illustrative purpose but there are task sets where the number of preemptions is even larger. If all \( T_i \) and \( C_i \) of tasks are multiplied by \( k \cdot \text{lcm} \) (where \( k \) is a positive integer) then the number of preemptions per job becomes at least \( k \cdot \text{lcm} \) times greater for \( PD^2 \). With this reasoning, it is easy to see that the number of preemptions divided by the number of jobs can approach infinity for \( PD^2 \). By multiplying \( C_i \) of all tasks by a constant such that \( U_s \) becomes 100%, we obtain that \( ER \) obtains an infinite number of preemptions per job as well.

All these three algorithms use the concept of lag of a task \( \tau_i \) at time \( t \). The lag is defined as \( \text{lag}(\tau_i,t)=t \cdot C_i/T_i \) allocated(\( \tau_i, [0,t) \)). (We let allocated(\( \tau_i, [0,t) \)) denote the amount of time that \( \tau_i \) was allocated during the time interval \([0,t)\)). The reason for the difference in the number of preemptions is that the constraints on \( \text{lag}(i,t) \) are different. \( PD^2 \) requires at every \( t \): \( -1<\text{lag}(i,t)<1 \). ER pfair requires at every \( t \): \( \text{lag}(i,t)<1 \) and bounded fairness requires at those \( t \) where a task arrives: \( -1<\text{lag}(i,t)<1 \).

We can see that relaxing the pfair constraint, as BF does, is useful in order to reduce the number of preemptions. In this example we assumed that \( T_i \) and \( C_i \) are integers; this is different from what we assume in the other parts of the paper and the reason for doing this is that pfair algorithms are only defined for the case where \( T_i \) and \( C_i \) are integer.
Figure 5. All non-preemptive scheduling algorithms have the utilization bound 0.

Figure 6. Pfair scheduling algorithms cause a large number of preemptions. The marks above the time line show the arrival times and deadlines of tasks. An arrow pointing upwards indicate an arrival time; an arrow pointing downwards indicate a deadline. The Gantt charts below the time line show the schedule of $PD^2$, Early-Release Pfair and Bounded fairness. For each scheduling algorithm, the Gantt chart shows the execution of each of the 6 tasks. A black filled box indicates that a task executes; a white box indicates that a task does not execute. Consequently, a black box followed by a white box indicates a context switch. If a job makes a context switch but if it has remaining execution, then it is a preemption.